Structure of non-equilibrium quantum phase transitions

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Motivation

✦ Equilibrium phase transitions (continuous):

- *Universal* - few critical exponents determine the physics (microscopic details irrelevant).
- *Spontaneous symmetry breaking* - in one phase the state does not possess the same symmetry as the full Hamiltonian.
- *Excitations* - energy gap closes at the critical point. Excitations in 'symmetry broken phase' either gapped (*Higgs*) or continuous (*Goldstone*).
- *Mermin-Wagner theorem* - the type of symmetry + the dimensionality determine which type of transition that is allowed.

✦ Non-equilibrium phase transitions: (Especially) cold atom experiments. Drive them and engineer desired coupling to reservoir $\longrightarrow$ non-trivial steady states.

✦ How do the above general results translate to these new phase transitions?

Outlook

1. (Equilibrium) Quantum phase transitions
   - Symmetry breaking.
   - Scale invariance and universality.
2. Open quantum systems
   - Composite systems, density operators.
   - Lindblad master equation.
3. Open quantum phase transitions
4. A new type of phase transition
5. Prospects

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Phase transitions

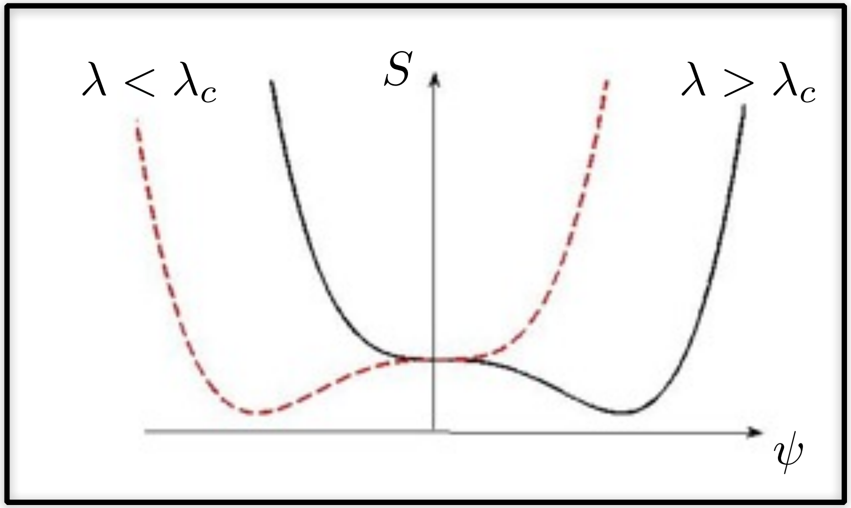
Phase transition

✦ Action $S[\bar{\psi},\psi]$, giving partition function

$$\mathcal{Z} = \int D(\bar{\psi},\psi)\, e^{-S[\bar{\psi},\psi]}$$

Mean-field solution: minimizing the action for *order parameter* ψ*.*

✦ PT$\longrightarrow$change in order parameter



Sudden jump - first order PT.

Phase transitions

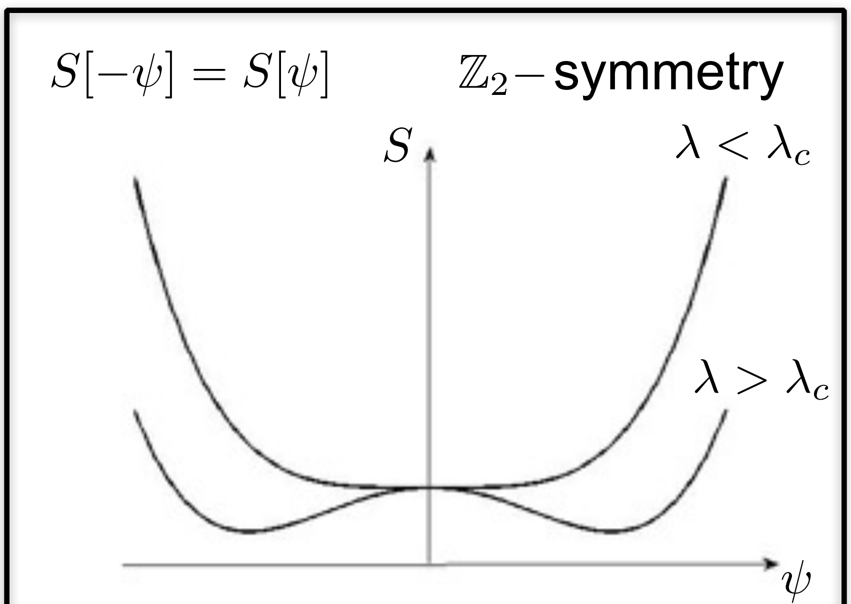
Phase transition

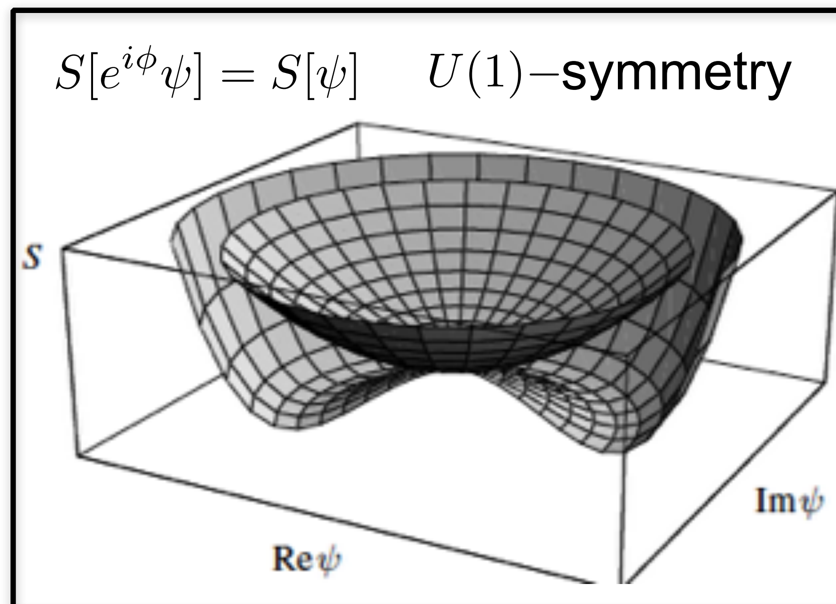
✦ Action $S[\bar{\psi}, \psi]$, giving partition function

$$\mathcal{Z} = \int D(\bar{\psi}, \psi)\, e^{-S[\bar{\psi}, \psi]}$$

Mean-field solution: minimizing the action for *order parameter* ψ*.*

✦ Symmetries $S[g\bar{\psi}, g\psi] = S[\bar{\psi}, \psi]$.





Quantum

Phase transitions

2nd order phase transition

- *Continuous* phase transition.
- Exists a symmetry $[\hat{U}, \hat{H}] = 0$.
- Quantum mechanics: $\hat{U}$ and $\hat{H}$ common eigenbasis - energy eigenstates well defined symmetry.
- Thermodynamic limit: *spontaneous symmetry breaking* - ground state $|\psi_0(\lambda)\rangle$ does not have a defined symmetry!
- "Potential barrier" becomes infinite $\longrightarrow$ degeneracy.
- If $\hat{U}$ continuous $\longrightarrow$ *Goldstone* (gapless) *excitations.*
  $\hat{U}$ *discontinuous* $\longrightarrow$ *Higgs* (gapped) *excitations.*

Quantum

Phase transitions

2nd order phase transition

✦ Ground state energy $E_0(\lambda)$ continuous, λ system parameter.

✦ Derivatives $\partial_\lambda^n E_0(\lambda)$ can be discontinuous for some *critical coupling* λ_c.

✦ When and why?

- *Thermodynamic limit* - system 'size' infinite.
- Competing terms supporting different properties

$$\hat{H} = \hat{H}_1 + \lambda \hat{H}_2, \qquad \left[\hat{H}_1, \hat{H}_2\right] \neq 0$$

✦ Roughly, $\lambda < 1$ ground state properties from $\hat{H}_1$, $\lambda > 1$ from $\hat{H}_2$.

- At $\lambda = \lambda_c \equiv 1$ spectrum gapless.
- $|\psi_0(\lambda)\rangle$ and ψ 'non-analytic' at λ_c.
- The PT driven by quantum fluctuations at $T = 0$ (classical PT, thermal fluctuations).

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Phase transitions

Phase transition (2nd) - scale invariance

At the critical point λ_c

The following Ising model configurations range from 2048x2048 sites to 131072x131072 sites.

Can you tell which is which?

1

$$H_{class} = \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i$$

System the same independent of “zooming” - length scale diverges!!

Phase transitions

Phase transition (2nd) - universality

✦ Systematic method (*renormalization group*) - eliminates short length scales ('high energies').

✦ Effective low energy model - microscopic details irrelevant.

✦ Models belong to different 'classes' - universality (depend on macroscopic properties like symmetries).

✦ Critical regime:

$$\begin{aligned} \xi &\propto |\lambda - \lambda_c|^{-\nu}, && \text{Characteristic length} \\ \tau &\propto |\lambda - \lambda_c|^{-\delta}, && \text{Characteristic time} \\ \Delta E &\propto |\lambda - \lambda_c|^{z\nu}, && \text{Gap closing} \\ &\ldots \end{aligned}$$

Critical exponents ν, δ, z, ... , different *universality classes*.

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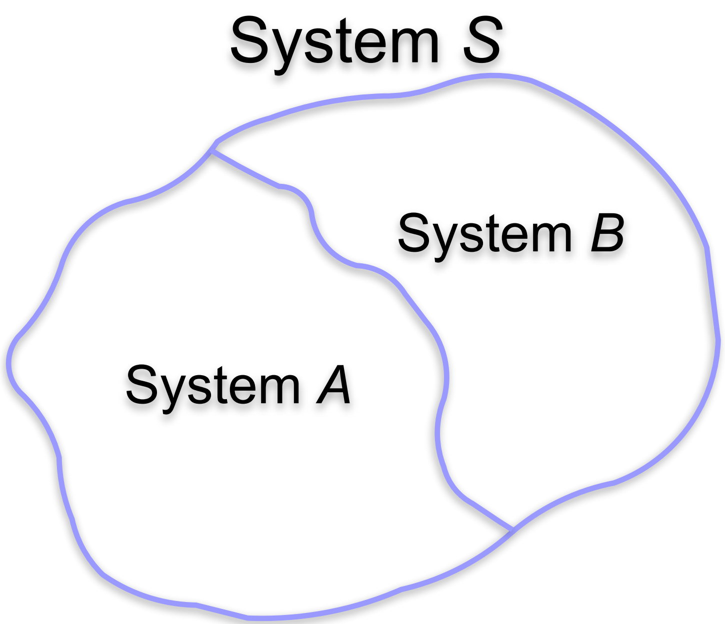
Open quantum systems
Density operator



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$|\psi\rangle = \sum_{ij} c_{ij} |\psi_i^A\rangle \otimes |\psi_j^B\rangle$$

- ✦ Only access to *A*, local observables $\hat{O}_A$.
- ✦ What is the state of subsystem *A*?

Open quantum systems
Density operator

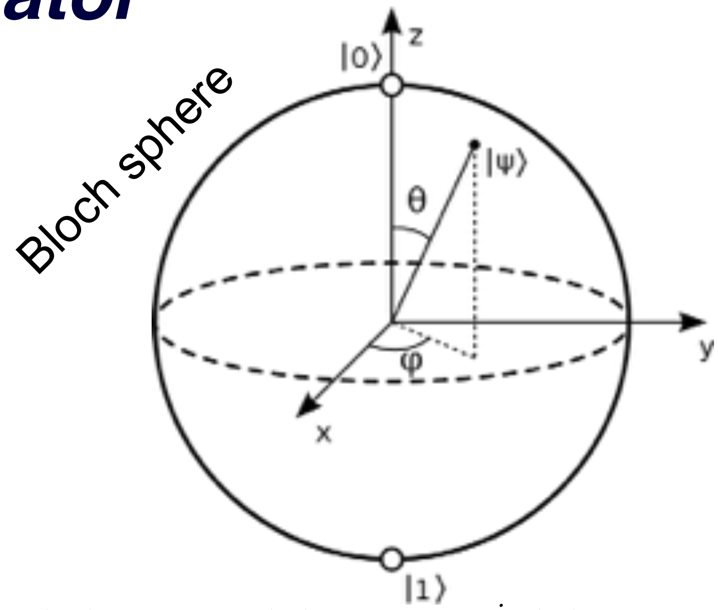
$$
\begin{aligned}
\langle \hat{\mathcal{O}}_A \rangle &= \textstyle\sum_{ijnm} c^*_{ij} c_{nm} \langle \phi^B_i | \langle \phi^A_j | \hat{\mathcal{O}}_A | \phi^A_n \rangle | \phi^B_m \rangle \\
&= \textstyle\sum_{ijn} c^*_{ij} c_{ni} \langle \phi^A_j | \hat{\mathcal{O}}_A | \phi^A_n \rangle \\
&= \textstyle\sum_l \sum_{ijn} c^*_{ij} c_{ni} \langle \phi^A_j | \hat{\mathcal{O}}_A | l \rangle \langle l | \phi^A_n \rangle \\
&= \textstyle\sum_l \sum_{ijn} c^*_{ij} c_{ni} \langle l | \phi^A_n \rangle \langle \phi^A_j | \hat{\mathcal{O}}_A | l \rangle \\
&= \textstyle\sum_l \langle l | \hat{\rho}_A \hat{\mathcal{O}}_A | l \rangle = \mathrm{Tr}_A \left[\hat{\rho}_A \hat{\mathcal{O}}_A \right]
\end{aligned}
$$

Reduced density operator $\hat{\rho}_A = \sum_{ijn} c^*_{ij} c_{ni} |\phi^A_n\rangle\langle\phi^A_j| = \mathrm{Tr}_B[\hat{\rho}]$, with $\hat{\rho} = |\psi\rangle\langle\psi|$.

State of system *S*: $\hat{\rho}$, state of subsystem *A*: $\hat{\rho}_A$. In general $\hat{\rho} \neq \hat{\rho}_A \otimes \hat{\rho}_B$.

Open quantum systems

Density operator



✦ In general $\hat{\rho} \neq |\psi\rangle\langle\psi|$.

✦ Physical:

$$\mathrm{Tr}\left[\hat{\rho}\right] = 1,$$
$$\hat{\rho}^{\dagger} = \hat{\rho},$$
$$||\hat{\rho}|| \geq 0$$

✦ If $\hat{\rho} = |\psi\rangle\langle\psi|$ (*pure state*), two-level state $|\psi\rangle = \cos\theta|0\rangle + \sin\theta e^{i\varphi}|1\rangle$.

✦ *Bloch vector* $\bar{R} = (x, y, z)$, $\alpha = \mathrm{Tr}\left[\hat{\sigma}_{\alpha}\hat{\rho}\right]$, with $\hat{\sigma}_{\alpha}$ *Pauli matrices.*

Pure state $|\bar{R}| = 1$, in general $|\bar{R}| < 1$.

Majority of states not pure!

✦ Random multi-qubit state $\hat{\rho} \longrightarrow$ reduced single qubit state $\hat{\rho}_1 \approx \mathbb{I}/2$.

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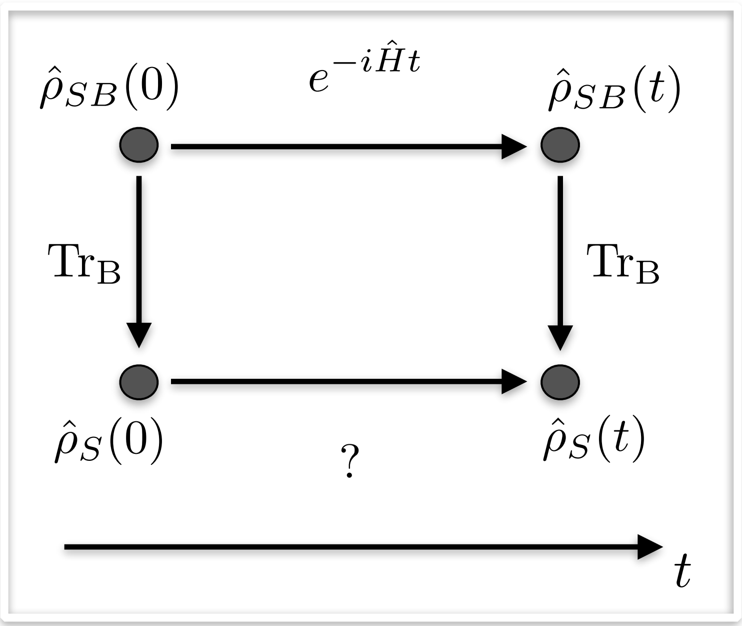
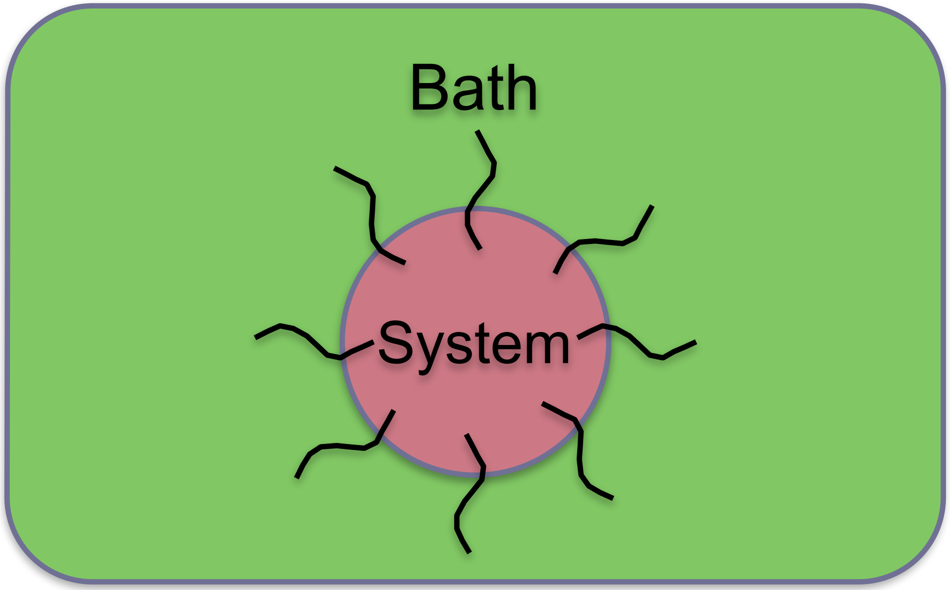
Open quantum systems
Density operator

Combined system

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB}$$





Generator "?", operators defined on $\mathcal{H}_S$. Generally **not** possible!!

Open quantum systems
Density operator

✦ Weak coupling, "big" bath

1. Bath time-scale short: no memory, *Markovian.*
2. System negligible influence on bath, *Born.*
3. *Rotating wave approximation.*

$$\partial_t \hat{\rho}(t) = i\left[\hat{\rho}(t), \hat{H}_S\right] + \hat{\mathcal{L}}\left[\hat{\rho}(t)\right],$$

$$\hat{\mathcal{L}}\left[\hat{\rho}(t)\right] = \sum_i g_i \left(2\hat{A}_i\hat{\rho}(t)\hat{A}_i^\dagger - \hat{A}_i^\dagger\hat{A}_i\hat{\rho}(t) - \hat{\rho}(t)\hat{A}_i^\dagger\hat{A}_i\right)$$

Lindblad master equation

$\hat{A}_i$ *Lindblad jump operators.* Non-unitary evolution, $\hat{\rho}(t)$ not generally pure.

Phase transitions in open quantum systems

- ✦ Quantum PT's, non-analyticity in ground state $|\psi_0(\lambda)\rangle$at critical coupling λ_c.
- ✦ Transition due to quantum fluctuations.
- ✦ Open quantum system

$$\partial_t \hat{\rho}(t) = i\left[\hat{\rho}(t), \hat{H}_S\right] + \hat{\mathcal{L}}\left[\hat{\rho}(t)\right],$$

$$\hat{\mathcal{L}}\left[\hat{\rho}(t)\right] = \sum_i g_i \left(2\hat{A}_i\hat{\rho}(t)\hat{A}_i^\dagger - \hat{A}_i^\dagger\hat{A}_i\hat{\rho}(t) - \hat{\rho}(t)\hat{A}_i^\dagger\hat{A}_i\right)$$

- ✦ **No** ground state, **no** energy spectrum!!

What do we mean by a PT here, what is the relevant state?

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Phase transitions in open quantum systems

✦ One (obvious) physically relevant state is the steady state

$$\partial_t \hat{\rho}_{ss}(t) = 0 \quad \rightarrow \quad i\left[\hat{\rho}_{ss}, \hat{H}_S\right] + \hat{\mathcal{L}}[\hat{\rho}_{ss}] = 0$$

✦ May be non-equilibrium, but time-independent.

✦ Closed system, all energy eigenstates also steady states.

✦ If:

1. $[\hat{A}_i, \hat{H}_S] = 0, \ \forall \hat{A}_i$, $\hat{A}_i$ hermitian, energy eigenstates also steady states.
2. $\hat{H}_S$ critical and $\hat{\mathcal{L}}\left[|\psi_0(\lambda < \lambda_c)\rangle\langle\psi_0(\lambda < \lambda_c)|\right] = 0$, the environment is expected to support the symmetric phase.

✦ If neither of the two $\longrightarrow$ new physics???

✦ Phase transition if $\hat{\rho}_{ss}$ non-analytic for some λ_c .

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New type of phase transition
Model

- ✦ Simplest (non-trivial) scenario
  - $\partial_t \hat{\rho} = i[\hat{\rho}, \hat{H}_s]$ trivial
  - $\partial_t \hat{\rho} = \kappa(2\hat{A}\hat{\rho}\hat{A}^\dagger - \hat{A}^\dagger\hat{A}\hat{\rho} - \hat{\rho}\hat{A}^\dagger\hat{A})$ trivial
  - $\partial_t \hat{\rho} = i[\hat{\rho}, \hat{H}_s] + \kappa(2\hat{A}\hat{\rho}\hat{A}^\dagger - \hat{A}^\dagger\hat{A}\hat{\rho} - \hat{\rho}\hat{A}^\dagger\hat{A})$ critical
- ✦ One such model (which is also solvable!)

$[\hat{S}_i, \hat{S}_j] = i\varepsilon_{ijk}\hat{S}_k$ Large spin-*S* (preserved)

$\hat{H}_S = \omega \hat{S}_x$ Coherent drive

$\hat{A} = \hat{S}_-$ Spontaneous decay to state $|S, -S\rangle$

New type of phase transition
Model

- Equation to solve

$$\partial\hat{\rho} = i[\hat{\rho}, \omega\hat{S}_x] + \kappa(2\hat{S}_-\hat{\rho}\hat{S}_+ - \hat{S}_+\hat{S}_-\hat{\rho} - \hat{\rho}\hat{S}_+\hat{S}_-)$$

- Limiting cases

$$\frac{\kappa}{\omega} = 0 \quad \rightarrow \quad |S, -S\rangle_x$$

Hamiltonian dominates

$$\frac{\omega}{\kappa} = 0 \quad \rightarrow \quad |S, -S\rangle_z$$

Lindbladian dominates

- Phase transition somewhere between?

New type of phase transition
Mean-field solution

- Mean-field (normal ordering): $S_\alpha = \langle \hat{S}_\alpha \rangle, \quad \langle \hat{S}_\alpha \hat{S}_\beta \rangle = \langle \hat{S}_\alpha \rangle \langle \hat{S}_\beta \rangle$.

$$\dot{S}_x = 2\frac{\kappa}{S} S_x S_z,$$

$$\dot{S}_y = S_z \left(2\frac{\kappa}{S} S_y - \omega \right),$$

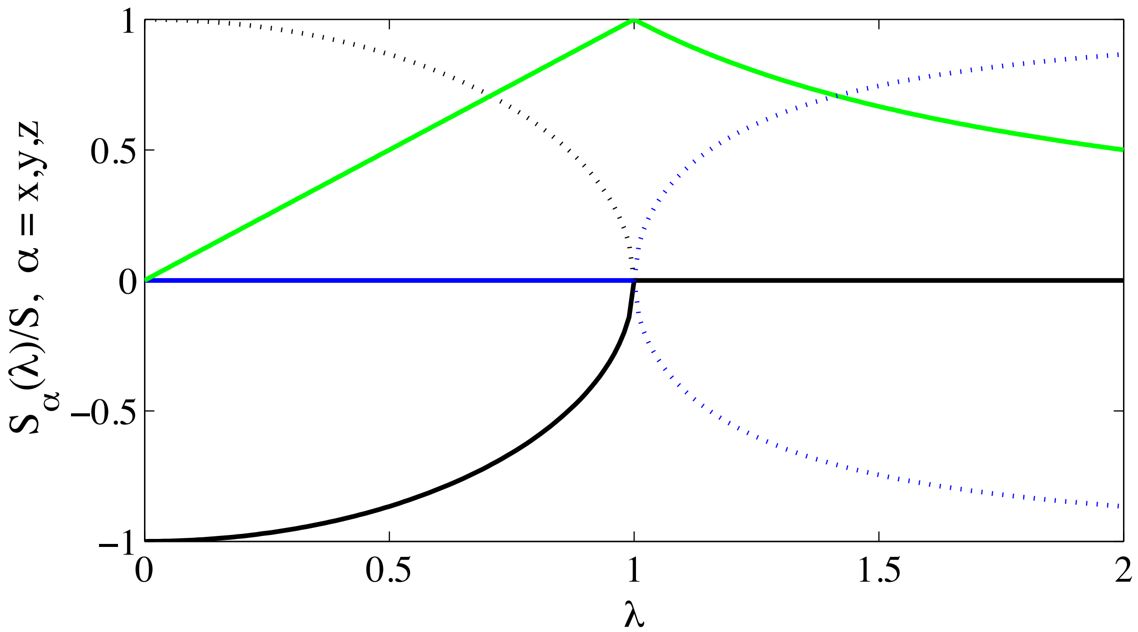
$$\dot{S}_z = \omega S_y - 2\frac{\kappa}{S} \left(S_x^2 + S_y^2 \right)$$

- Conserved spin: $S^2 = S_x^2 + S_y^2 + S_z^2$.

- Steady state solutions $\lambda = \dfrac{\omega}{2\kappa}$

$$(S_x,\, S_y,\, S_z) = \left(\pm\sqrt{1 - \frac{1}{\lambda}}, \frac{1}{\lambda}, 0 \right), \qquad (S_x,\, S_y,\, S_z) = \left(0, \lambda, \pm\sqrt{1 - \lambda} \right).$$

New type of phase transition
Mean-field solution



Classical steady state solutions: *x* (blue), *y* (green), *z* (black).

Blue curve: *Hopf* (like) *bifurcation* (stability, purely imaginary eigenvalues).

Black curve: *Strange bifurcation* (lower branch stable, upper unstable).

New type of phase transition

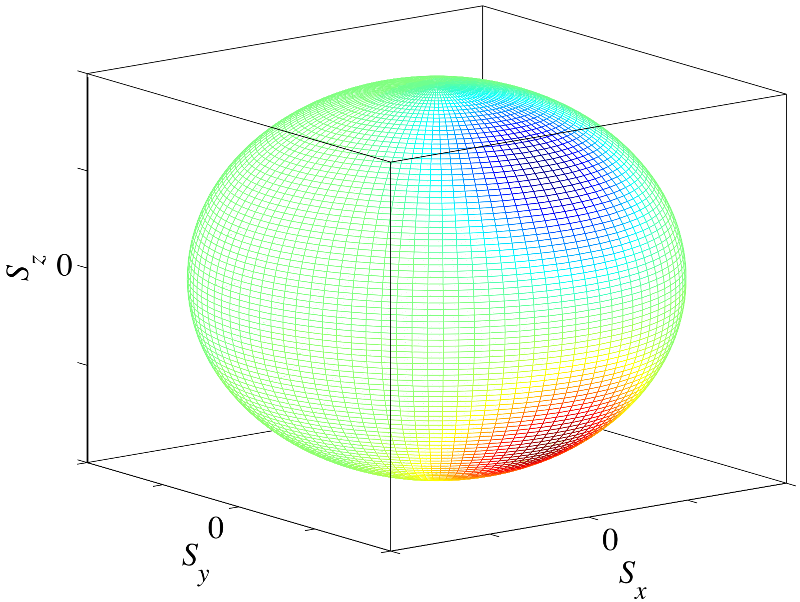
Mean-field solution

- ✦ Understanding the bifurcation?
- ✦ Rewrite eom's in canonical variables $\phi,\ z = \cos\theta$

$$\dot{z} = -2\kappa\left(1 - z^2\right) + \omega\sqrt{1 - z^2}\,\sin\phi,$$

$$\dot{\phi} = -\omega\frac{z}{\sqrt{1 - z^2}}\cos\phi$$

- ✦ Not possible to assign a (local) 'potential' for these
- ✦ Phase space = sphere, think of it as one attractive and one repulsive fixed point.
- ✦ Not possible on a plane.



New type of phase transition
Universality - mean-field

✦ Universality - critical exponents.

- "Magnetisation" $S_z \propto |\lambda_c - \lambda|^{1/2} \longrightarrow \nu = 1/2$.

  (Quantum $\nu = 1/2$)

✦ 'Critical slowing down'. Linear stability around the mean-field solution - eigenvalues gives relaxation time

$$T \propto |\lambda_c - \lambda|^{-1/2}, \qquad \beta = 1/2$$

✦ Seems to be universal!

New type of phase transition
Universality - quantum

✦ Quantum treatment. Analytical solution

$$\hat{\rho}_{ss} = \frac{1}{D}\sum_{n,m=0}^{2S}(g^*)^{-m}g^{-n}\hat{S}_-^m\hat{S}_+^n, \qquad g = i\omega S/\kappa$$

✦ Expectations $\langle\hat{\mathcal{O}}\rangle = \mathrm{Tr}[\hat{\mathcal{O}}\hat{\rho}_{ss}]$. Truncate for some *S*.

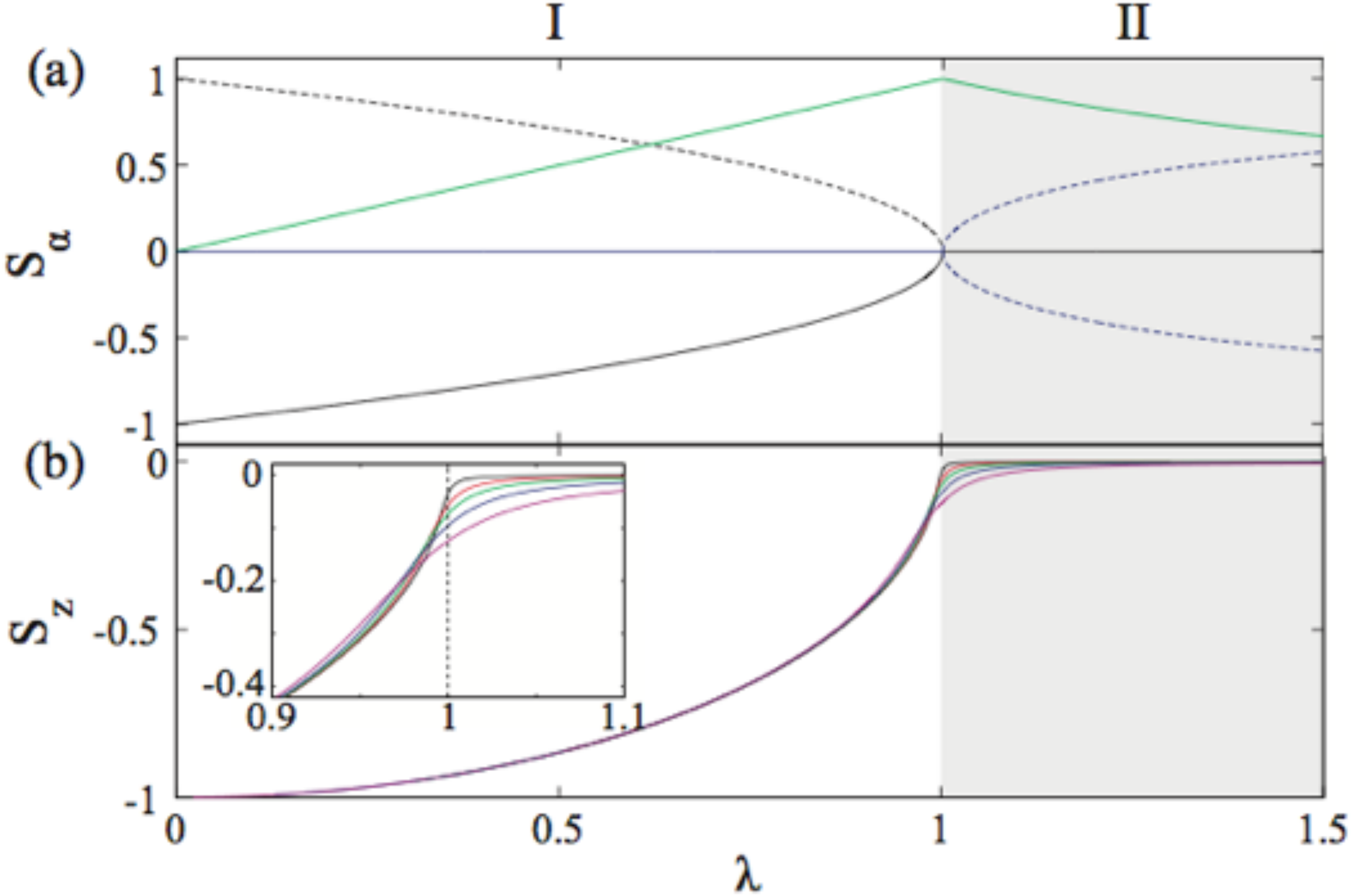
✦ For continuous PT we demand continuous expectations of ‘local’ operators

$$\hat{\mathcal{O}} = \sum_i \hat{o}_i$$

$\hat{o}_i$ single qubit operator

New type of phase transition
Universality - quantum

✦ Magnetisation (local) critical exponent the same as for mean-field.



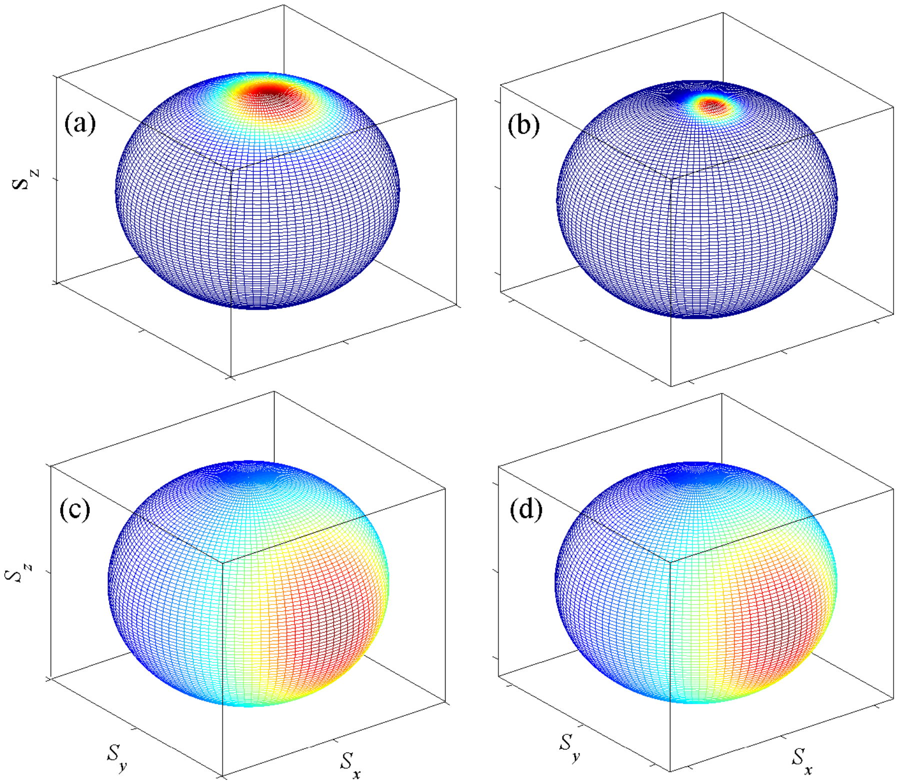
New type of phase transition
Universality - quantum

- ✦ $\langle \hat{S}_z \rangle / S \to 0, \;\; \omega/\kappa \to \infty$. Hamiltonian part dominates.
- ✦ $\langle \hat{S}_z \rangle / S \to 1, \;\; \omega/\kappa \to 0$. Pure state, fluctuations $\langle \hat{S}_x^2 \rangle / S^2 \to 0,$
- ✦ In general, quantum fluctuations relative system size $\mathcal{O}(S^{-1})$.
- ✦ But $\Delta S_\alpha^2 / S^2 \to 1/3, \;\; \omega/\kappa \to \infty$! Reservoir-induced-fluctuations.

New type of phase transition
Universality - quantum

✦ Fluctuations reflected in the phase space distribution.

$$Q(z) = \langle z|\hat{\rho}_{\rm ss}|z\rangle$$ Husimi Q-function



Magnetized phase

Incoherent phase

New type of phase transition
Symmetries

✦ Symmetries.

- *Closed case*. $[\hat{A}, \hat{H}] = 0 \leftrightarrow \hat{H} = \hat{U}\hat{H}\hat{U}^{-1} \longrightarrow$ $\hat{A}$ conserved.
- *Open case*. $\partial\hat{\rho} = \mathcal{L}_{\hat{L}}[\hat{\rho}] = \mathcal{L}_{\hat{U}\hat{L}\hat{U}^{-1}}[\hat{\rho}] \longrightarrow$ generally no conserved quantity.

✦ Guess $\hat{S}_x \to -\hat{S}_x,\; \hat{S}_y \to \hat{S}_y,\; \hat{S}_z \to -\hat{S}_z$

$$\partial\hat{\rho} = i[\hat{\rho}, \omega\hat{S}_x] + \kappa\left(2\hat{S}_-\hat{\rho}\hat{S}_+ - \hat{S}_+\hat{S}_-\hat{\rho} - \hat{\rho}\hat{S}_+\hat{S}_-\right)$$

$$\longrightarrow$$

$$\partial\hat{\rho} = -i[\hat{\rho}, \omega\hat{S}_x] + \kappa\left(2\hat{S}_+\hat{\rho}\hat{S}_- - \hat{S}_-\hat{S}_+\hat{\rho} - \hat{\rho}\hat{S}_-\hat{S}_+\right)$$

Dual symmetry! Flip spectrum + “step up” instead of “step down”.

New type of phase transition

Second order phase transition with no symmetry breaking!!

✦ Corresponding Hamiltonian system (drive + 'internal' energy)

$$\hat{H} = \omega \hat{S}_x + \kappa \hat{S}_z$$

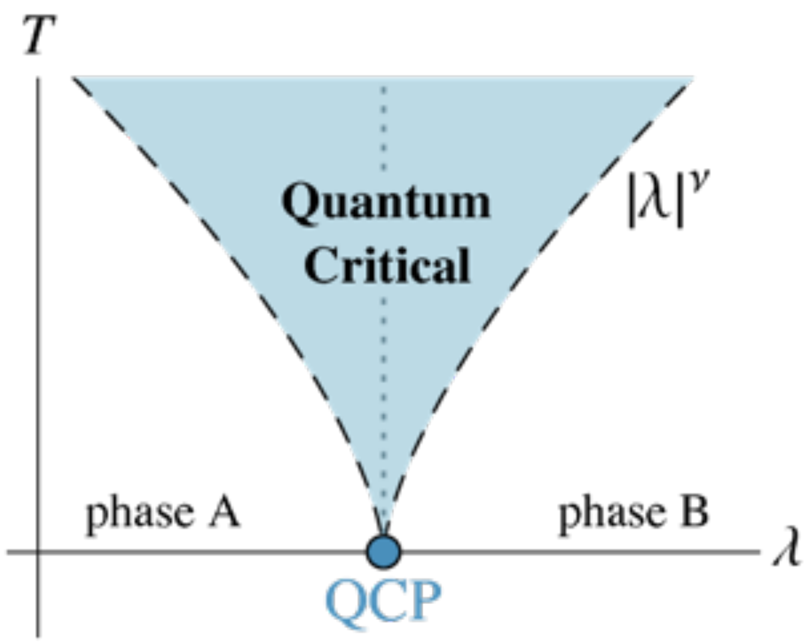
clearly not critical.

✦ Interplay between 'bath noise' and unitary evolution.

New type of phase transition
Quantum or classical

✦ Is it a quantum phase transition?

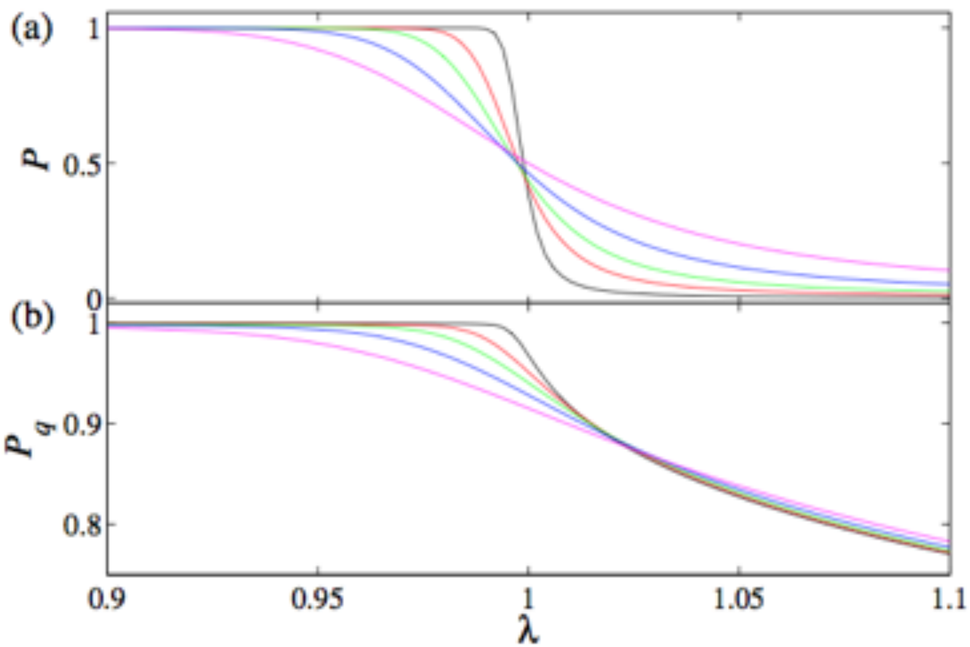
- Single degree of freedom. Quantum fluctuations vanishingly small in thermodynamic limit.
- Fluctuations due to coupling to reservoir.
- *Quantum critical region* - signatures of “quantum” survives finite temperatures in the vicinity of a critical point.



New type of phase transition
Quantum or classical

✦ "Quantum"

- Coherence: *Purity* $P \equiv \mathrm{Tr}[\hat{\rho}^2] > 0$ (thermodynamic limit).
- Entanglement: *Negativity* $N(\hat{\rho})$ between pair of qubits.



Thermodynamic limit:

(a), full system purity. Jump!! Extensive operator, not local.

(b) Single qubit purity. Pure state in magnetised phase!

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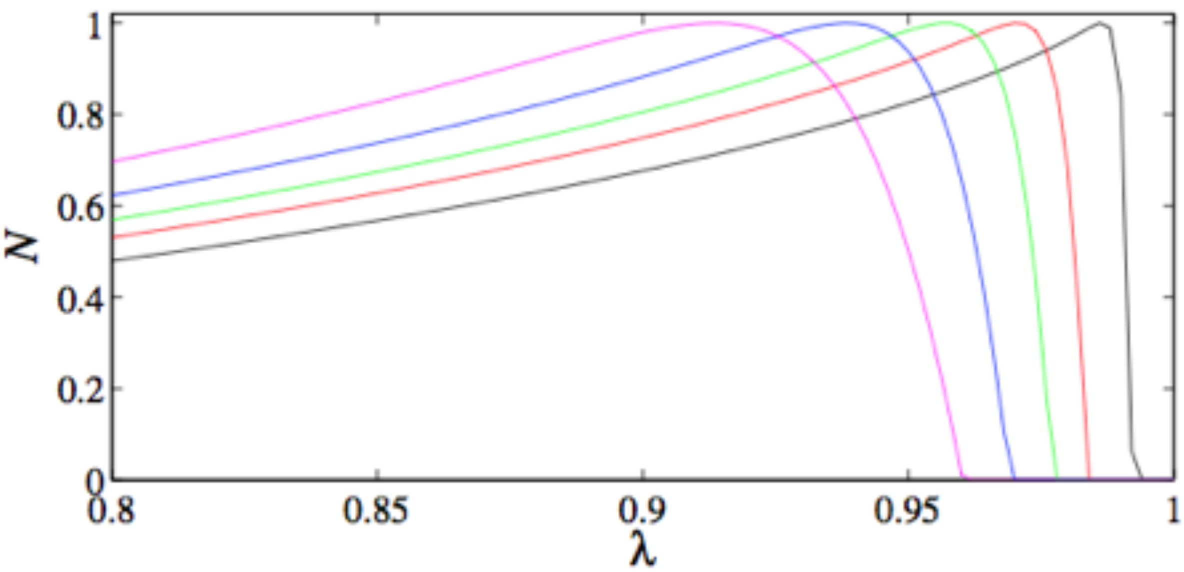

<u>Thermodynamic limit</u>:

Scaled negativity

$$\max(\mathcal{N}) = 1$$

Entanglement vanishes ($\sim S^{-0.9}$).

Peak at critical point.

New type of phase transition
Quantum or classical

✦ "Quantum"

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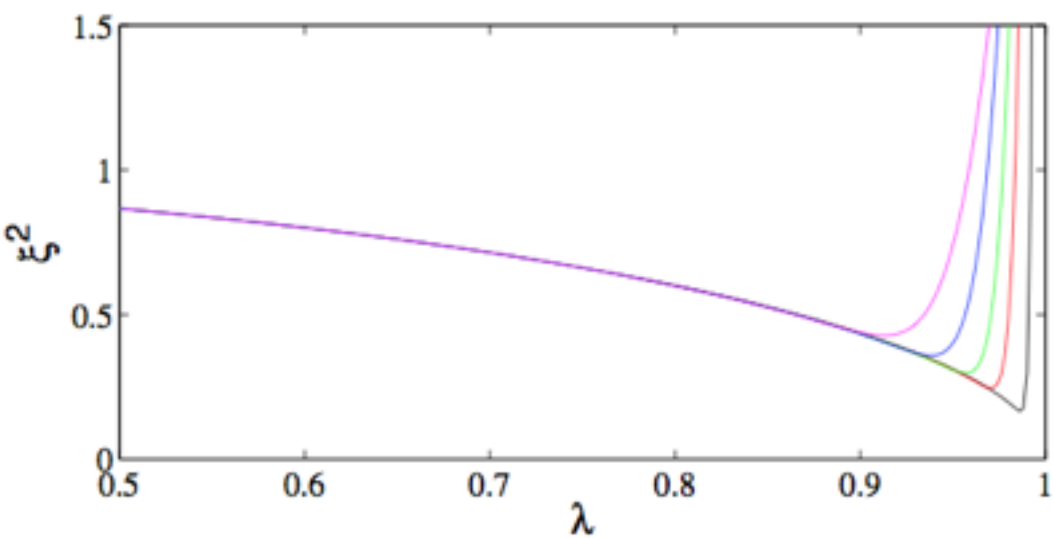
- Entanglement: *Negativity* $N(\hat{\rho})$ between pair of qubits.



Thermodynamic limit:

Spin squeezing for different system sizes.

Entanglement witness. Global entanglement.

'Entanglement sharing'.

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Phase transitions in open quantum systems

✦ Found a model possessing an open PT, seemingly second order but lacks any symmetry breaking $\longrightarrow$ **New type**.

Open questions

- Are these PT's universal? Scale invariance? New classes?
- Models driven more strongly by quantum fluctuations, length scale (lattice models).
- Gap closening of the *Liouvillian*, universal?
- Is there a *Mermin-Wagner theorem* for open QPT's?
- *Kibble-Zurek mechanism*?


Thank you!