Homework problems, set 6:

Quantum circuits

Hand in before 23/5-2018.
Maximum number of points: 37.

1 Quantum gates?

Which of the following two-qubit gates are unitary?

a) $|a\rangle \otimes |b\rangle \to |b\rangle \otimes |\bar{a}\rangle$. (2p)

b) $|a\rangle \otimes |b\rangle \to |\bar{a}\rangle \otimes |a\rangle$. (2p)

c) $|a\rangle \otimes |b\rangle \to |a \oplus b\rangle \otimes |a\rangle$. (2p)

d) $|a\rangle \otimes |b\rangle \to \frac{1}{\sqrt{2}}\Big(|a\rangle \otimes |b\rangle + |\bar{a}\rangle \otimes |\bar{b}\rangle\Big)$. (2p)

e) $|a\rangle \otimes |b\rangle \to (-1)^{a+b}|a\rangle \otimes |b\rangle$. (2p)

f) $|a\rangle \otimes |b\rangle \otimes |c\rangle \to |a\rangle \otimes |b\rangle \otimes |c \oplus a \wedge b\rangle$. (2p)

Here $\oplus$ represents addition modulo 2.

g) Pick one of the above gates (one that's unitary and not the last one!) and draw a quantum circuit realising the operation. (3p)

(The operation in f is called the *Toffoli gate*.)

2 Quantum correspondences of Boolean gates

Assume that we want to directly generalise the Boolean operations NOT, AND, and XOR to quantum gates.

a) Prove that there exist no universal quantum NOT gate, *i.e.* a unitary $\hat{U}_{\text{NOT}}$ such that
$$\hat{U}_{\text{NOT}}|\psi\rangle = |\psi_\perp\rangle, \tag{1}$$
where $|\psi\rangle$ is a general pure qubit state and $\langle\psi|\psi_\perp\rangle = 0$. (2p)

b) A quantum AND gate could be imagined as

$$\hat{U}_{\text{AND}}|A\rangle|B\rangle = |A\rangle|A \wedge B\rangle, \tag{2}$$

where A and B are either 0 or 1. Prove that such a gate is not allowed quantum mechanically. (2p)

c) Similarly, the quantum XOR gate would be

$$\hat{U}_{\text{XOR}}|A\rangle|B\rangle = |A\rangle|A \oplus B\rangle. \tag{3}$$

Show that this gate is indeed possible. (2p)

3 Universal gates

Which of the following sets of gates is universal?

i) Single-qubit gates and CS gates.

ii) Single-qubit gates and swap gates.

iii) Single-qubit gates and CZ gates.

In the computational basis, the CS gate is diag$\{1, 1, 1, i\}$, and the CZ gate diag$\{1, 1, 1 - 1\}$. (4p)

(Hint, you might try using these combinations of gates to construct a CNOT gate.)

4 The CNOT gate

In the computational basis, the CNOT gate can be represented by the matrix

$$\hat{U}_{\text{CNOT}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4}$$

a) Calculate each of the states generated if the target qubit is initially in an eigenstate of $\hat{\sigma}_z$ and the control bit in an eigenstate of $\hat{\sigma}_x$. (3p)

b) Assume that you have access to single-qubit gates and one CNOT, what is the the most general control-unitary gate you can construct? (3p)

5 Large numbers

The speed-up of quantum computations, compared to classical ones, lies in the intrinsic parallelism, *i.e.* single (quantum) operations can transform a large number of numbers simultaneously. Furthermore, it is well known that storing large amount of data on a classical computer demands much memory. Assume now that we consider n-qubit states $\hat{\rho}$.

a) For some general state $\hat{\rho}$, how many real numbers do we need to describe it? (2p)

b) If we know that the state $\hat{\rho}$ is pure, how many numbers do we then need in order to describe it? (2p)

c) Finally, assume we know that we have a product state $\hat{\rho} = \bigotimes_{l=1}^{n} \hat{\rho}^{(l)}$ (thus, the state is disentangled), where $\hat{\rho}^{(l)}$ is the state of qubit l. How many real numbers do we then need to give the most general product state? (2p)