Homework problems, set 5:

State discrimination

Hand in before 16/5-2018.
Maximum number of points: 34.

1 Fidelity and purity

Any qubit density operator can be written

$$\hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \vec{R}\cdot\vec{\sigma}\right) = \frac{1}{2}\left(\mathbb{1} + R_x\hat{\sigma}_x + R_y\hat{\sigma}_y + R_z\hat{\sigma}_z\right), \tag{1}$$

where $\vec{R}$ is the Bloch vector. The square root of a density operator, $\hat{\varrho} = (\hat{\rho})^{1/2}$ (i.e. $\hat{\varrho}^2 = \hat{\rho}$), can be written

$$\hat{\varrho} = \frac{1}{2}\left(\zeta\mathbb{1} + r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z\right). \tag{2}$$

Moreover, the general expression for the fidelity between two states $\hat{\rho}_1$ and $\hat{\rho}_2$ is given by

$$F(\hat{\rho}_1, \hat{\rho}_2) = \mathrm{Tr}\left[\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}}\right]. \tag{3}$$

a) Find the eigenvalues λ_1 and λ_2 of $\hat{\rho}$. Also, calculate the determinant $\det(\hat{\rho})$. (2p)

b) Express ζ, r_x, r_y and r_z in the parameters R_x, R_y and R_z. Is $\hat{\varrho}$ a density operator? (3p)

c) For a pure state we can parametrize the density operator according to

$$\hat{\rho} = \frac{1}{2}\left[\mathbb{1} + \sin 2\theta\cos\varphi\hat{\sigma}_x - \sin 2\theta\sin\varphi\hat{\sigma}_y + \left(\cos^2\theta - \sin^2\theta\right)\hat{\sigma}_z.\right] \tag{4}$$

Show that the purity $\mathrm{Tr}\left[\hat{\rho}^2\right] = 1$ and that the von Neumann entropy $S_{vN}(\hat{\rho}) = -\lambda_1\log\lambda_1 - \lambda_2\log\lambda_2 = 0$ for a pure state (4). (3p)

d) For two qubit states $\hat{\rho}_1$ and $\hat{\rho}_2$ it can be shown that the fidelity can be calculated as

$$F(\hat{\rho}_1, \hat{\rho}_2) = \left(\mathrm{Tr}\left[\hat{\rho}_1\hat{\rho}_2\right] + 2\sqrt{\det(\hat{\rho}_1)\det(\hat{\rho}_2)}\right)^{1/2}. \tag{5}$$

Give an expression for the fidelity of two qubit states parametrised according to Eq. (1). Given that the two states are pure, *i.e.* $\hat{\rho}_i = |\psi_i\rangle\langle\psi_i|$ for $i = 1, 2$, what is the fidelity in this case? (3p)

2 Fidelity and independence of representation

In an euclidian vector space, rotations are given by orthogonal transformations (symmetric matrices, *i.e.* real, with unit determinant). For example, the scalar product between two euclidian vectors is preserved under such transformations (this is to say that the angle between vectors is invariant under rotations). Likewise, the scalar product between two pure states in quantum mechanics (*i.e.* the fidelity) is preserved under unitary transformations. Show that this holds also for mixed states, that is

$$F\left(\hat{U}\hat{\rho}_1\hat{U}^{-1}, \hat{U}\hat{\rho}_2\hat{U}^{-1}\right) = F\left(\hat{\rho}_1, \hat{\rho}_2\right) \tag{6}$$

for some unitary operator $\hat{U}$. (4p)

3 Qutritt trace distance

For qubits we have that the *Hilbert-Schmidt distance*

$$||\hat{\rho}_1, \hat{\rho}_2|| = \sqrt{\mathrm{Tr}\left[(\hat{\rho}_1 - \hat{\rho}_2)^2\right]/2} \tag{7}$$

equals the *trace distance*

$$D\left(\hat{\rho}_1, \hat{\rho}_2\right) = \frac{1}{2}\mathrm{Tr}\left[||\hat{\rho}_1 - \hat{\rho}_2||\right]. \tag{8}$$

Is the same true for qutrits? (4p)

4 Trace preserving maps and distances I

If $\hat{\mathcal{A}}$ is a trace preserving operator, that is $\mathrm{Tr}\left[\hat{\mathcal{A}}\left(\hat{\rho}\right)\right] = \mathrm{Tr}\left[\hat{\rho}\right]$, we have learned that

$$\begin{aligned} D\left(\hat{\mathcal{A}}(\hat{\rho}_1), \hat{\mathcal{A}}(\hat{\rho}_2)\right) &\leq D\left(\hat{\rho}_1, \hat{\rho}_2\right), \\ F\left(\hat{\mathcal{A}}(\hat{\rho}_1), \hat{\mathcal{A}}(\hat{\rho}_2)\right) &\geq F\left(\hat{\rho}_1, \hat{\rho}_2\right), \end{aligned} \tag{9}$$

where $D\left(\hat{\rho}_1, \hat{\rho}_2\right)$ and $F\left(\hat{\rho}_1, \hat{\rho}_2\right)$ are the trace distance and the fidelity respectively.

a) Prove or disprove that (4p)

$$\begin{aligned} D\left(\hat{\mathcal{A}}(\hat{\rho}_1), \hat{\rho}_2\right) &\leq D\left(\hat{\rho}_1, \hat{\rho}_2\right), \\ F\left(\hat{\mathcal{A}}(\hat{\rho}_1), \hat{\rho}_2\right) &\geq F\left(\hat{\rho}_1, \hat{\rho}_2\right). \end{aligned} \tag{10}$$

Consider now the composite three-qubit states

$$\begin{aligned} |\phi^{(1)}_{ABC}\rangle &= |W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + [010\rangle + |001\rangle), \\ |\phi^{(2)}_{ABC}\rangle &= |GHZ\rangle = \frac{1}{\sqrt{2}}(|100\rangle + [011\rangle). \end{aligned} \tag{11}$$

b) For the states $\hat{\rho}^{(i)}_{ABC} = |\phi^{(i)}_{ABC}\rangle\langle\phi^{(i)}_{ABC}|$, $\hat{\rho}^{(i)}_{AB} = \mathrm{Tr}_C\left[\hat{\rho}^{(i)}_{ABC}\right]$, and $\hat{\rho}^{(i)}_{A} = \mathrm{Tr}_B\left[\hat{\rho}^{(i)}_{AB}\right]$, with $i = 1, 2$, calculate the corresponding trace distances and fidelities (for example $F(\hat{\rho}^{(1)}_A, \hat{\rho}^{(2)}_A)$). Comment on the result; how it relates to Eq. (9). (4p)

5 Trace preserving maps and distances II

A qubit evolving under the influence of Markovian *phase damping* is described by the master equation

$$\frac{d}{dt}\hat{\rho} = i\left[\hat{\rho}, \hat{\sigma}_3\right] - \frac{\gamma}{2}\left(\hat{\rho} - \hat{\sigma}_3\hat{\rho}\hat{\sigma}_3\right), \qquad \gamma \geq 0. \tag{12}$$

Let us represent the density operator as

$$\hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \vec{R}\cdot\vec{\sigma}\right) = \frac{1}{2}\left(\mathbb{1} + R_x\,\hat{\sigma}_1 + R_y\,\hat{\sigma}_2 + R_z\,\hat{\sigma}_3\right), \tag{13}$$

where $\vec{R} = (R_x, R_y, R_z)$ is the Bloch vector.

a) Solve the master equation for the initial conditions $(R_x(0), R_y(0), R_z(0)) = (1, 0, 0)$ and $(R_x(0), R_y(0), R_z(0)) = (\cos\theta, \sin\theta, 0)$ for any $0 < \theta < \pi/2$. If the two solutions are represented by the states $\hat{\rho}_1(t)$ and $\hat{\rho}_2(t)$, calculate $D\left(\hat{\rho}_1, \hat{\rho}_2\right)$ and $F\left(\hat{\rho}_1, \hat{\rho}_2\right)$ and comment how the results relate to Eq. (9). (You may use the solutions derived in the previous homework set.) (5p)