Homework problems, set 4:

Quantum and classical communication

Hand in before 4/5-2018.
Maximum number of points: 34.
(Some problems might need simple numerical calculations...)

1 Quantum copy machine

Following the book *Quantum approach to informatics* we look at the example of "quantum copying" on pages 40-41.

a) Write down a unitary operator $\hat{U}$, acting on the three qubit states $|a\rangle_1|b\rangle_2|c\rangle_3$, which realises the transformation (2.102). Check that $\hat{U}$ is unitary. (3p)

b) From the state $|\Psi\rangle_{123}$ of equation (2.103), derive the three reduced density operators ρ_α, $\alpha = 1, 2, 3$. (3p)

2 Noisy classical channel

It is by no means necessary for a channel to have the same number of input and output symbols. Consider a channel with two input symbols (a_1, a_2) and four output symbols (b_1, b_2, b_3, b_4). The conditional probabilities are

$$\begin{aligned} P(b_1|a_1) = P(b_2|a_1) &= \frac{1}{3}, \\ P(b_3|a_2) = P(b_4|a_2) &= \frac{1}{3}, \\ P(b_3|a_1) = P(b_4|a_1) &= \frac{1}{6}, \\ P(b_1|a_2) = P(b_2|a_2) &= \frac{1}{6}. \end{aligned} \tag{1}$$

a) Calculate the associated mutual information

$$I(A:B) = H(B) - H(B|A). \tag{2}$$

Express it in the probabilities $\{p_1, p_2\}$. (4p)

b) Calculate the channel capacity, *i.e.* the maximum value of $I(A:B)$. (3p)

3 Quantum binary channel

We consider communication between Alice and Bob. Assume that Alice's coding states are

$$\begin{aligned} |a_0\rangle &= |e_1\rangle, \\ |a_1\rangle &= \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle), \end{aligned} \tag{3}$$

and that Bob's detection states are

$$\begin{aligned} |b_0\rangle &= \cos\theta|e_1\rangle - \sin\theta|e_2\rangle, \\ |b_1\rangle &= \sin\theta|e_1\rangle + \cos\theta|e_2\rangle. \end{aligned} \tag{4}$$

During the lecture we looked at the symmetric case, *i.e.* $P_{00} = P_{11}$, corresponding to $\theta = \pi/8$. Let us lift the restriction of a symmetric channel.

a) If we can vary the detection angle θ and the probabilities $\{p_0, p_1\}$ (of transmitting the two states $|a_1\rangle$ and $|a_1\rangle$ respectively) and we want to maximise the channel capacity, how should we choose these? (4p)

4 Strong subadditivity

During the lectures we proved that

$$|S(\hat{\rho}_A) - S(\hat{\rho}_B)| \leq S(\hat{\rho}_{AB}) \leq S(\hat{\rho}_A) + S(\hat{\rho}_B). \tag{5}$$

The first inequality is called the *triangle inequality* or the *Araki-Lieb inequality*, and the second *subadditivity*. For three-partite systems (subsystems A, B and C) one can prove the so called *strong subadditivity*

$$S(\hat{\rho}_{ABC}) + S(\hat{\rho}_B) \leq S(\hat{\rho}_{AB}) + S(\hat{\rho}_{BC}) \tag{6}$$

a) Use the "strong subadditivity" inequality of Eq. (6) to prove the inequalities (5). (4p)

b) Consider now the GHZ and W states

$$\begin{aligned} |\Psi_{GHZ}\rangle &= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \\ |\Psi_W\rangle &= \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \end{aligned} \tag{7}$$

Show by first calculating the reduced density operators and then the corresponding von Neumann entropies that the strong subadditivity is fulfilled. (3p)

5 Negativity

Given a general (bi-partite) density operator $\hat{\rho} \in \mathcal{H}_A \otimes \mathcal{H}_B$;

$$\hat{\rho} = \sum_{ijkl} c_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|, \tag{8}$$

the *partial transpose* of say subsystem B is given by

$$\hat{\rho}^{T_B} = \sum_{ijkl} c_{ijkl} |i\rangle\langle j| \otimes |l\rangle\langle k|. \tag{9}$$

For two qubits, for example, this would imply

$$\begin{aligned} |x,0\rangle\langle y,1| &\to |x,1\rangle\langle y,0|, \\ |x,1\rangle\langle y,0| &\to |x,0\rangle\langle y,1|. \end{aligned} \tag{10}$$

Furthermore, if λ_i are the eigenvalues of $\hat{\rho}^{T_B}$, the *negativity* is defined as

$$\mathcal{N}[\hat{\rho}] = \frac{1}{2}\sum_i |\lambda_i| - \lambda_i. \tag{11}$$

Thus, it sums all negative eigenvalues. For qubits, a non-zero value of $\mathcal{N}(\hat{\rho})$ is a necessary and sufficient condition for entanglement between the two qubits. Note that this measure of entanglement is valid also for mixed states. The von Neumann entropy, for example, is not a good measure of entanglement for mixed two-qubit states.

Assume that two qubits interact via the Hamiltonian

$$\hat{H} = \hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)}. \tag{12}$$

a) Find a closed form for the time-evolution operator (3p)

$$\hat{U}(t) = e^{-i\hat{H}t}. \tag{13}$$

(Hint, remember how you derived the time-evolution operator for a single spin.)

b) Give the time evolved states if they initially are in the states (3p)

$$\begin{aligned} &(i) \quad \hat{\rho}(0) = \tfrac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right), \\ &(ii) \quad \hat{\rho}(0) = \tfrac{1}{3}\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |01\rangle\langle 01| + i|00\rangle\langle 11| - i|11\rangle\langle 00|\right). \end{aligned} \tag{14}$$

c) Calculate the negativity $\mathcal{N}[\hat{\rho}(t)]$ for the two cases above. (4p)