Homework problems, set 3:

Nonlocality

Hand in before 20/4-2018.
Maximum number of points: 44.

1 Entanglement

Characterizing *multipartite entanglement* becomes very complex in higher dimensions. For three qubits, however, is it possible to show that there exists two classes, W and GHZ, *i.e.* any entangle (pure) three quit state can be obtained from local operations of these two states

$$\begin{aligned} |\Psi\rangle_W &= \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle), \\ |\Psi\rangle_{GHZ} &= \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \end{aligned} \tag{1}$$

Let us call the three subsystems A, B and C.

a) Calculate the reduced density operators $\hat{\rho}_A^{(W)} = \mathrm{Tr}_{BC}\left[\hat{\rho}^{(W)}\right]$ and $\hat{\rho}_A^{(GHZ)} = \mathrm{Tr}_{BC}\left[\hat{\rho}^{(GHZ)}\right]$. With the obtained reduced density operators, calculate the corresponding *von Neumann entropy* (4p)

$$\begin{aligned} S_A^{(W)} &= -\mathrm{Tr}_A\left[\hat{\rho}_A^{(W)}\log\left(\hat{\rho}_A^{(W)}\right)\right], \\ S_A^{(GHZ)} &= -\mathrm{Tr}_A\left[\hat{\rho}_A^{(GHZ)}\log\left(\hat{\rho}_A^{(GHZ)}\right)\right]. \end{aligned} \tag{2}$$

b) Repeat the exercise above but for the reduced densities $\hat{\rho}_{BC}^{(W)} = \mathrm{Tr}_A\left[\hat{\rho}^{(W)}\right]$ and $\hat{\rho}_{BC}^{(GHZ)} = \mathrm{Tr}_A\left[\hat{\rho}^{(GHZ)}\right]$. (3p)

c) Having the expressions for the reduced density operators $\hat{\rho}_{BC}^{(W)}$ and $\hat{\rho}_{BC}^{(GHZ)}$ from the previous problem, calculate the corresponding *neg-*

ativity for the two states

$$\mathcal{N}\left[\hat{\rho}\right] = \sum_i \frac{|\lambda_i| - \lambda_i}{2}, \tag{3}$$

where the λ_i's are the eigenvalues of the *partially transposed* density operators $\left(\hat{\rho}_{BC}^{(W)}\right)^{T_B}$ and $\left(\hat{\rho}_{BC}^{(GHZ)}\right)^{T_B}$. (4p)

Note. Remember that for a (bi-partite) density operator $\hat{\rho} \in \mathcal{H}_A \otimes \mathcal{H}_B$;

$$\hat{\rho} = \sum_{ijkl} c_{ijkl} |i\rangle\langle j| \otimes |k\rangle\langle l|, \tag{4}$$

the *partial transpose* of say subsystem B is given by

$$\hat{\rho}^{T_B} = \sum_{ijkl} c_{ijkl} |i\rangle\langle j| \otimes |l\rangle\langle k|. \tag{5}$$

2 Entropic density of entangled qubit states

The von Neumann entropy for a qubit state $0 \leq S \leq 1$ provided we use log-base 2.

a) Assume you pick a random qubit state $\hat{\rho}$ and calculate its corresponding entropy S. Further, assume that all qubit states are equally probable (i.e. they are drawn from a flat distribution). Plot the corresponding distribution $P(S)$ which gives the probability that the state you picked has exactly entropy S. (2p)

b) Now let us change scenario and assume that we pick instead a random pure n-qubit state

$$|\psi\rangle = \sum_{\{\sigma\}} C_{\{\sigma\}} |\{\sigma\}\rangle. \tag{6}$$

Here the sum is over all 2^n possible states $|\sigma_1, \sigma_2, \ldots, \sigma_n\rangle$ with $\sigma_i = 0,\, 1$. Having $|\psi\rangle$ we can trace out all but one qubit and get a single qubit state $\hat{\rho}_n$, which in return has some entropy S_n. Like for the a) exercise we can introduce the corresponding distribution $P(S_n)$. Nothing says that $P(S_n)$ will be independent of the number of qubits n. Plot $P(S_n)$ for $n = 2$ and some other $n > 2$. (3p)

c) Argue what the distribution $P(S_n)$ is in the limit $n \to \infty$, and discuss what it implies physically - what can we say about general multi-qubit states? (2)

(If you're going to hand in this set of problems in time I suggest you use numerics to solve this one!)

3 POVM's

During the lecture we discussed an example demonstrating the *Naimark's theorem.* We considered a qubit and the POVM operators $\hat{\Pi}_n = |\Psi_n\rangle\langle\Psi_n|$;

$$\begin{aligned}
|\Psi_1\rangle &= \frac{1}{\sqrt{2}}(\tan\theta|0\rangle + |1\rangle), \\
|\Psi_2\rangle &= \frac{1}{\sqrt{2}}(\tan\theta|0\rangle - |1\rangle), \\
|\Psi_3\rangle &= \sqrt{1-\tan^2\theta}|0\rangle.
\end{aligned} \tag{7}$$

In the extended space we instead consider the projectors $\hat{P}_n = |\Phi_n\rangle\langle\Phi_n|$;

$$\begin{aligned}
|\Phi_1\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle\otimes|0\rangle + \tan\theta|0\rangle\otimes|0\rangle + \sqrt{1-\tan^2\theta}|1\rangle\otimes|1\rangle\right), \\
|\Phi_2\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle\otimes|0\rangle - \tan\theta|0\rangle\otimes|0\rangle - \sqrt{1-\tan^2\theta}|1\rangle\otimes|1\rangle\right), \\
|\Phi_3\rangle &= \sqrt{1-\tan^2\theta}|0\rangle\otimes|0\rangle - \tan\theta|1\rangle\otimes|1\rangle, \\
|\Phi_4\rangle &= |0\rangle\otimes|1\rangle.
\end{aligned} \tag{8}$$

a) Show that the two set of operators are complete, *i.e.* $\sum_{n=1}^{3}\hat{\Pi}_n = \mathbb{1}$ and $\sum_{n=1}^{4}\hat{P}_n = \mathbb{1}$. (3p)

b) Show orthonormality; $\hat{P}_n\hat{P}_m = \hat{P}_n\delta_{nm}$. (2p)

c) Show that $(\langle 0|\otimes\langle\psi|)\hat{P}_n(|\psi\rangle\otimes|0\rangle) = \langle\psi|\hat{\Pi}_n|\psi\rangle$ for $n = 1, 2, 3$. What is $(\langle 0|\otimes\langle\psi|)\hat{P}_4(|\psi\rangle\otimes|0\rangle)$? (3p)

4 Master equations I

In the lecture we considered the master equation

$$\frac{d}{dt}\hat{\rho} = i\left[\hat{\rho}, \hat{a}^\dagger \hat{a}\right] - \frac{C}{2}\left(\hat{a}^\dagger \hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger \hat{a}\right) - \frac{A}{2}\left(\hat{a}\hat{a}^\dagger \hat{\rho} - 2\hat{a}^\dagger \hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger\right). \quad (9)$$

a) Show that the norm is preserved, *i.e.* $d\mathrm{Tr}[\hat{\rho}]/dt = 0$. (3p)

b) Let us define the *quadratures*

$$\hat{x} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right), \qquad \hat{p} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right). \quad (10)$$

Use the result

$$\frac{d}{dt}\langle\hat{a}\rangle = \left(-i - \frac{C-A}{2}\right)\langle\hat{a}\rangle \quad (11)$$

derived in the book, to give expressions for $d\langle\hat{x}\rangle/dt$ and $d\langle\hat{p}\rangle/dt$. Solve these equations of motion and describe how the state evolves in phase space, *i.e.* in the xp-plane, for the initial condition $(\langle\hat{x}\rangle_0, \langle\hat{p}\rangle_0) = (x_0, 0)$. Do the solutions converge to a steady state? (3p)

5 Master equation II

A qubit evolving under the influence of Markovian *phase damping* is described by the master equation

$$\frac{d}{dt}\hat{\rho} = i\left[\hat{\rho}, \hat{\sigma}_3\right] - \frac{\gamma}{2}\left(\hat{\rho} - \hat{\sigma}_3\hat{\rho}\hat{\sigma}_3\right), \qquad \gamma \geq 0. \quad (12)$$

Furthermore, any qubit state can be written as

$$\hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \vec{R}\cdot\vec{\sigma}\right) = \frac{1}{2}\left(\mathbb{1} + u\,\hat{\sigma}_1 + v\,\hat{\sigma}_2 + w\,\hat{\sigma}_3\right), \quad (13)$$

where $\vec{R} = (u, v, w)$ is the Bloch vector.

a) Write down the equation-of-motion for the Bloch vector $\vec{R}$. Use the knowledge of the previous exercise to solve these equations. Describe how the Bloch vector evolves on the Bloch sphere for the initial states $(u(0), v(0), w(0)) = (1, 0, 0)$ and $(u(0), v(0), w(0)) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$ for any $0 \leq \phi \leq \pi$ and $0 < \theta < \pi/2$. (4p)

b) For a general *Lindblad master equation*

$$\frac{d}{dt}\hat{\rho} = i\left[\hat{\rho}, \hat{H}\right] + \mathcal{L}\left[\hat{\rho}\right], \tag{14}$$

where the Lindblad *super-operator* $\mathcal{L}[\hat{\rho}]$ describes the coupling to the reservoir/bath, a *dark state* $\hat{\rho}_D$ is defined as an 'eigenstate' of $\hat{H}$, *i.e.* $\left[\hat{\rho}_D, \hat{H}\right] = 0$, and also an eigenstate of the Lindblad super operator with eigenvalue 0, *i.e.* $\mathcal{L}[\hat{\rho}_D] = 0$. The set $\{\hat{\rho}_D\}$ of all dark states is called a *decoherence-free subset*. Find this set for the Lindblad master equation (12). (3p)

c) Describe what happens to the diagonal and off-diagonal terms of the density matrix $\hat{\rho}(t)$ as time progresses. (Assume $\hat{\rho}(t)$ to be given in the computational basis.) (2p)

d) If we would instead consider the master equation

$$\frac{d}{dt}\hat{\rho} = i\left[\hat{\rho}, \hat{\sigma}_3\right] - \frac{\gamma}{2}\left(\hat{\rho} - \hat{\sigma}_1\hat{\rho}\hat{\sigma}_1\right). \tag{15}$$

What would be the infinite time ($t = \infty$) density operator $\hat{\rho}(\infty)$, *i.e.* its steady state? Can you determine the decoherence-free subset $\{\hat{\rho}_D\}$? (3p)