Homework problems, set 2:

Composite systems

Hand in before 11/4-2018.
Maximum number of points: 29.

1 Density operators

Let us parametrize the pure qubit state with the angles θ and φ;

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)\, e^{i\varphi}|1\rangle, \tag{1}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. A general state can then be written as

$$\hat{\rho} = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \, \sin\theta P(\theta, \varphi)|\psi\rangle\langle\psi|. \tag{2}$$

a) What constrains do the probability $P(\theta, \varphi)$ (i.e. $P(\theta, \varphi) \geq 0$) have to fulfil in order for $\hat{\rho}$ to be a proper density operator? (2p)

b) If $P(\theta, \varphi)$ is uniformly distributed over the entire Bloch sphere, can you write down the state $\hat{\rho}$ as a 2×2 matrix? What is the corresponding Bloch vector? (2p)

2 Operators

Assume that the operators $\hat{A}$ and $\hat{B}$ are defined on a Hilbert space $\mathcal{H}$ (that is, if $|\psi\rangle \in \mathcal{H}$, then $\hat{A}|\psi\rangle$ and $\hat{B}|\psi\rangle \in \mathcal{H}$) with dimension n.

a) Prove the *cyclic property* $\mathrm{Tr}\left[\hat{A}\hat{B}\right] = \mathrm{Tr}\left[\hat{B}\hat{A}\right]$, where the trace is complete over $\mathcal{H}$. (2p)

b) Using $\mathrm{Tr}\left[\hat{A}\hat{B}\right] = \mathrm{Tr}\left[\hat{B}\hat{A}\right]$, prove that the trace is unique, *i.e.* it is independent on basis choice. (2p)

c) Prove the *linearity property* $\mathrm{Tr}\left[a\hat{A}+b\hat{B}\right]=a\mathrm{Tr}\left[\hat{A}\right]+b\mathrm{Tr}\left[\hat{A}\right]$, where $a,\, b\in\mathbb{C}$. (2p)

d) Prove the equality $\det\left[\exp(\hat{A})\right]=\exp(\mathrm{Tr}\hat{A})$, for a diagonalizable operator $\hat{A}$. (2p)

e) Assume you have a composite system, $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$ with $\mathcal{H}_A$ and $\mathcal{H}_B$ finite dimensional, and two operators that can be decomposed $\hat{O}_1=\hat{A}_1\otimes\hat{B}_1$ and $\hat{O}_2=\hat{A}_2\otimes\hat{B}_2$. Prove or disprove the equality $\mathrm{Tr}_A\left[\hat{O}_1\hat{O}_2\right]=\mathrm{Tr}_A\left[\hat{O}_2\hat{O}_1\right]$. (2p)

3 Purification

Let $\hat{\rho}_A$ be the state for system S_A, and ρ_i and $|\phi_i\rangle$ its corresponding eigenvalues and eigenstates. Construct an ON-basis $\{|\varphi_j\rangle\}$ for a system S_B and the state

$$|\Psi\rangle=\sum_i\sqrt{\rho_i}|\phi_i\rangle\otimes|\varphi_i\rangle. \tag{3}$$

a) If $\hat{A}$ is an operator defined for subsystem S_A, prove that (2p)

$$\mathrm{Tr}_A\left[\hat{A}\hat{\rho}_A\right]=\langle\Psi|\hat{A}\otimes\mathbb{1}|\Psi\rangle. \tag{4}$$

b) Is this 'purification' unique, i.e. can you find another state $|\tilde{\Psi}\rangle$ that also equals $\mathrm{Tr}_A\left[\hat{A}\hat{\rho}_A\right]$? (2p)

4 Purity

The maximum value of the purity $P(\hat{\rho})=\mathrm{Tr}\left[\hat{\rho}^2\right]$ is unity, but what is the minimum value of the purity, and what is the corresponding density operator? (5p)

Hint: You may use *Lagrange multipliers*.

5 Qutritt

Naturally, the quantum information carrier does not necessarily have to be a qubit, if it is composed of three possible quantum states $|0\rangle$, $|1\rangle$ and $|2\rangle$ it is called a qutritt. In the last set of homework problems we introduced the *Gell-Mann matrices*

$$\hat{\lambda}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\lambda}_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \hat{\lambda}_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{5}$$

$$\hat{\lambda}_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and said that they span the set of 3×3 matrices. Thus, any qutritt state can be written

$$\hat{\rho} = \frac{1}{3}\mathbb{1} + \frac{1}{\sqrt{3}}\vec{\mathbf{R}} \cdot \vec{\lambda}, \tag{6}$$

where $\vec{\mathbf{R}} = (r_1,\, r_2,\, \dots,\, r_8)$ and $\vec{\lambda} = (\hat{\lambda}_1,\, \hat{\lambda}_2,\, \dots,\, \hat{\lambda}_8)$. It thereby follows that the qutritt 'Bloch vector' has eight components and the 'Bloch sphere' is a seven dimensional surface.

a) Given $\hat{\rho}$, how would you calculate the Bloch vector $\vec{\mathbf{R}}$? (2p)

b) In terms of the Bloch vector, calculate the purity $P(\hat{\rho}) = \mathrm{Tr}\,[\hat{\rho}^2]$. (2p)

c) Prove that for a general (real) $\vec{\mathbf{R}}$, $\hat{\rho}$ does not necessarily need to be a density operator. This fact makes the geometry of the set of qutritt states much more complicated than that of qubits (regular Bloch sphere). (2p)