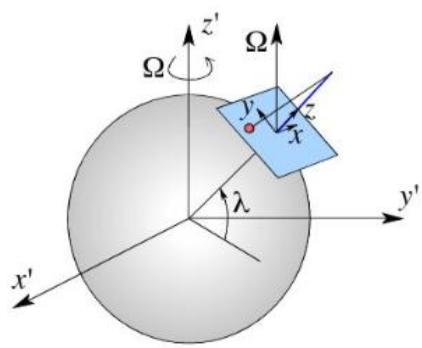
The Foucault pendulum

Old problem demonstrating the rotation of earth.



1 Problem

This is an old experiment invented by Foucault to demonstrate that the earth is rotating. The idea of this assignment is to use the techniques of analytical mechanics to explain how the Foucault pendulum works. Below I include a problem with solution taken from a book. The problem is divided into numerous steps. How you wish to structure your report is up to you, you do not need to follow the same steps as in the book, nor do you need to include all of them (that will be difficult within 4 pages). However, your report should explain how with the pendulum it is possible to predict the earth rotation. And you should use what you have learned in the course, preferable Lagrangian mechanics.

1. ~~Write down the Lagrangian of the system. If x increases by the arbitrary quantity s, what is the increase in y in order for the potential to be unchanged ? Deduce that the Lagrangian is invariant in a group of oblique translations.~~

2. ~~Write down the new set of coordinates, which depend continuously on s, and which leave the Lagrangian invariant. With the help of Noether's theorem show that the quantity $\dot{x} + \frac{1}{2}\dot{y}$ is a constant of the motion.~~

   ~~*Check* – This result can be checked writing the two Lagrange equations and eliminating the derivative of the potential.~~

2.4. Foucault's Pendulum

[Solution and Figure p. 79] ⋆ ⋆

Study of a famous experiment in the Lagragian formalism

This experiment was realized, in March 31st 1851, with a 67 m pendulum beneath the dome of the Pantheon; it was revived in 1902, after Foucault's death (1819–1868), by Camille Flammarion (1842–1925).

Let a simple pendulum of length l and with mass m be located at a latitude λ (complementary angle between the vertical at this point and the Earth's rotation axis $\boldsymbol{Z}$) on the Earth surface. The pendulum thus moves on a sphere (Fig. 2.2). One wishes to study the effect of the Earth's rotation on the motion of the pendulum, in a very elegant way, using the Lagrangian formalism. The effect due to the Earth's revolution around the Sun is neglected.

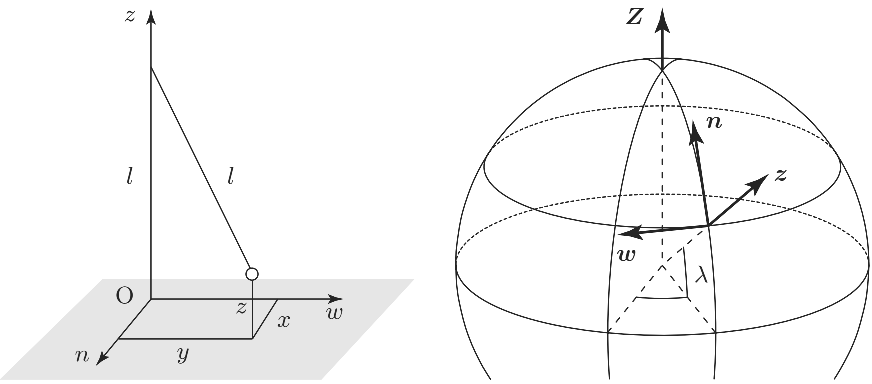


Fig. 2.2 Foucault's pendulum. On the left hand side, the trajectory of Foucault's pendulum on the ground. On the right hand side, the position of the pendulum on the Earth's surface

We will take as generalized coordinates x, y the deviations with respect to the vertical along the south-north direction $\boldsymbol{n}$ and along the east-west direction $\boldsymbol{w}$.

1. In the limit of **small deviations** with respect to the vertical, show that the potential energy to within a constant, is

$$V(x,y) \cong mg\frac{x^2+y^2}{2l}.$$

   Here g represents the acceleration due to gravity only, the direction of which is the true vertical.

2. Give, with respect to the northerly axis $\boldsymbol{n}$, the westerly axis $\boldsymbol{w}$ and the true vertical (passing through the Earth's center) $\boldsymbol{z}$ at this position, the components of the pendulum velocity with respect to the Earth. One neglects $\dot{z}$. Why?

3. Give the components of the unit vector $\boldsymbol{Z}$ (rotation axis of the Earth, see Fig. 2.2) on $\boldsymbol{n}$ and $\boldsymbol{z}$.

4. Give the driving velocity due to the Earth's rotation (radius R_E, angular velocity Ω). The term in z is neglected. Why?

5. Derive the expression of the kinetic energy and of the Lagrangian. One neglects all terms containing the square of the Earth's rotational speed, except those containing the Earth's radius. Justify.

6. Write down the Lagrange equations.

7. The equation for the x coordinate contains a constant term. Show that it can be eliminated by the substitution $\tilde{x} = x - x_e$. Give the value of the constant x_e and its interpretation.

8. To solve the two Lagrange equations on an equal footing, the complex function $u(t) = \tilde{x}(t) + i\,y(t)$ is introduced. Show that the equation to be solved is

$$\ddot{u} + 2i\Omega\,\sin\lambda\,\dot{u} + gu/l = 0.$$

   Rather than using a systematic method for finding the solution, it is convenient to make the change of function $u(t) = U(t)\,\exp(irt)$.

   Choose r and ω as real numbers in order to obtain the equation $\ddot{U} + \omega^2 U = 0$.

   What is the nature of the motion seen in terms of variables X, Y with $U = X + i\,Y$?

   Describe the motion in terms of variables x, y?

   What happens at the pole and on the equator?

Hints:

- The velocity of a point M, which is part of a solid and which rotates with angular velocity Ω around the axis $\boldsymbol{u}$ is $\Omega \boldsymbol{u} \times \boldsymbol{OM}$ where O is any point on the rotation axis.
- In the complex plane, the multiplication by $\exp(irt)$ rotates a point by an angle rt. This method is known as the switch towards rotating axes. It is also employed for a charged particle in a magnetic field.
- The equilibrium position is not the vertical at the point under consideration (there exists also the centrifugal force which acts against gravity).

Alternative derivation

One works in the non Galilean frame $\boldsymbol{n}$, $\boldsymbol{w}$, $\boldsymbol{z}$.

Express the Coriolis force (the centrifugal force is neglected).

Find the potential corresponding to the Coriolis force?

Write down the Lagrangian and the Lagrange equations.

Compare with the first method.

The Lagrangian is time-independent. What is the constant of the motion?

~~2.5. Three-particle System~~

~~**[Solution and Figure p. 82]**~~ ⋆ ⋆

~~*How astute changes of variables allow us to exhibit symmetries*~~

~~A – Changing coordinates~~

~~Let us consider a system formed with three particles, of equal mass m, constrained to move on a straight line $x'Ox$. They interact via a potential that depends only on the relative distance between them. This system can represent the vibrations of a linear triatomic molecule.~~

~~Let q_1, q_2, q_3, the abscissae, be chosen as generalized coordinates. The corresponding Lagragian is then written:~~

$$\cancel{L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - V_1(q_2 - q_3) - V_2(q_3 - q_1) - V_3(q_1 - q_2).}$$

~~1. Give the constant of the motion, associated with spatial translations.~~

2. ~~Noether's theorem can be applied safely. There exists a constant of the motion~~

$$\cancel{I = (\partial_{\dot{x}} L)\, (dX/ds)|_0 + (\partial_{\dot{y}} L)\, (dY/ds)|_0}$$

or: $$I = \dot{x} + \frac{\dot{y}}{2} = \text{ const.}$$

~~One can directly check this property starting from the two Lagrange's equations $m\ddot{x} = -V'$, $m\ddot{y} = 2V'$, and eliminating the potential derivative to get $\ddot{x} + \frac{1}{2}\ddot{y} = 0$ which, after integration, gives again the result of~~ Noether's theorem.

2.4. Foucault's Pendulum

[Statement and Figure p. 59]

1. Let us adopt the axis conventions represented in Fig. 2.2. In the laboratory rotating frame, the point of suspension of the pendulum lies at altitude l. At equilibrium, the pendulum mass is placed at the origin. For a small deviation θ with respect to the vertical, the altitude is $z = l(1 - \cos\theta) \approx l\theta^2/2$. Let us perform an approximate calculation of the gravitational potential energy: $V = mgz \approx (mgl\theta^2)/2$. Furthermore,

$$\theta^2 \approx \sin^2\theta = \frac{OM^2}{l^2} = \frac{x^2 + y^2}{l^2}.$$

This leads to an approximate expression for the gravitational potential:

$$V(x, y) = mg\frac{x^2 + y^2}{2l}.$$

2. The pendulum coordinates in the frame attached to the Earth are by definition (x, y, z). The pendulum velocity in this frame is thus $(\dot{x}, \dot{y}, \dot{z})$. For a small deviation from equilibrium, the coordinates x and y are of order $l\theta$, whereas z is of order $l\theta^2$, hence negligible with respect to the horizontal components. It is then fully justified to consider that the motion takes place in the horizontal plane and that the relative velocity is given by:

$$\boldsymbol{v}_r = \dot{x}\,\boldsymbol{n} + \dot{y}\,\boldsymbol{w}.$$

3. The vector $\boldsymbol{n}$ is in the plane defined by the pole axis and the true vertical $\boldsymbol{z}$. As a consequence, the unit vector along the pole axis $\boldsymbol{Z}$ is in the plane formed by the vectors $(\boldsymbol{n}, \boldsymbol{z})$. A simple analysis based on the various projections shows that:

$$\boldsymbol{Z} = \cos\lambda\,\boldsymbol{n} + \sin\lambda\,\boldsymbol{z}.$$

4. The instantaneous rotation vector is directed along the pole axis: $\boldsymbol{\Omega} = \Omega \boldsymbol{Z}$. Let M_0 represent the coincident pendulum point at a given time. The driving velocity is simply expressed as $\boldsymbol{v}_e = \boldsymbol{\Omega} \times \boldsymbol{OM}_0$. Using the definition

$$\boldsymbol{OM}_0 = x\,\boldsymbol{n} + y\,\boldsymbol{w} + (R_E + z)\,\boldsymbol{z}$$

and the previous relation to express the instantaneous rotation vector, one obtains the equation which gives the driving velocity:

$$\boldsymbol{v}_e = -\Omega y \sin\lambda\,\boldsymbol{n} + \Omega\,(x \sin\lambda - (R_E + z)\cos\lambda)\;\boldsymbol{w} + \Omega y \cos\lambda\,\boldsymbol{z}. \quad (2.14)$$

5. The absolute velocity of the pendulum in the Galilean frame is obtained by summing the relative velocity and the driving velocity given in Questions 2 and 4: $\boldsymbol{v}_a = \boldsymbol{v}_r + \boldsymbol{v}_e$. The kinetic energy is obtained from $T = \frac{1}{2} m \boldsymbol{v}_a^2$. Taking into account the small value $\Omega \approx 10^{-5} rad/s$, values around unity for x, y , the very small value of z and the very large value of $R_E \approx 10^6$ m, one must retain in the expression for T the terms Ωx, Ωy and $R_E \Omega^2$ (order 10^{-5}) but one can neglect the terms $\Omega^2 x^2$, $\Omega^2 y^2$, Ωz and $\Omega^2 xz$ (order 10^{-10}). Lastly, the Lagrangian L is the difference between the kinetic energy T and the potential energy as given in Question 1. It is of the form:

$$L = \frac{1}{2} m \left[\dot{x}^2 + \dot{y}^2 - 2\Omega \dot{x} y \sin\lambda + 2\Omega \dot{y}(x \sin\lambda - R_E \cos\lambda)\right.$$
$$\left. - R_E \Omega^2 x \sin(2\lambda)\right] - \frac{m\tilde{g}(x^2 + y^2)}{2l}$$

up to an uninteresting constant $m\Omega^2 R_E^2 \cos^2\lambda/2$ and in which we introduced the effective gravitational field $\tilde{g} = g - \Omega^2 R_E \cos^2\lambda$ modified by the centrifugal force.

6. We are concerned now by the Lagrange equations giving the motion in the horizontal plane. Starting from the previous Lagrangian and applying the traditional recipe (2.4), one obtains the equations of motion:

$$\ddot{x} - 2\Omega \dot{y} \sin\lambda + \frac{\tilde{g}}{l} x + \frac{1}{2} R_E \Omega^2 \sin(2\lambda) = 0;$$
$$\ddot{y} + 2\Omega \dot{x} \sin\lambda + \frac{\tilde{g}}{l} y = 0.$$

7. Let define $\tilde{x} = x - x_e$ and substitute this value in the first Lagrange equation; the arbitrary value x_e is then chosen in order to cancel the constant term in the resulting equation. Owing to the fact that the value $R_E \Omega^2 \approx 10^{-3}\,m/s^2$ is very small as compared to $g \approx 10\,m/s^2$, it

is legitimate to approximate $\tilde{g} \cong g$, in which case the result takes the following form:

$$x_e = -\frac{R_E l \Omega^2 \sin(2\lambda)}{2g}.$$

The set $\tilde{x} = 0, y = 0$ is a solution of the equations of motion; indeed this is the equilibrium solution. Thus at equilibrium, the pendulum is not oriented along the true vertical, but along the apparent vertical, which makes an angle

$$\alpha \approx \sin\alpha = \frac{x_e}{l} = R_E \Omega^2 \sin\frac{2\lambda}{2g}$$

with respect to the true vertical. This deviation is due to the centrifugal force. It is maximum on the 45th parallel.

8. The coupled differential equations to be solved are rewritten:

$$\ddot{\tilde{x}} - 2\Omega\dot{y}\sin\lambda + \frac{\tilde{g}\tilde{x}}{l} = 0;$$
$$\ddot{y} + 2\Omega\dot{\tilde{x}}\sin\lambda + \frac{\tilde{g}y}{l} = 0.$$

Let us introduce the complex variable $u = \tilde{x} + iy$. Multiply the second Lagrange equation by i and add the first one; the auxiliary variable u occurs naturally in the unique differential equation:

$$\ddot{u} + 2i\Omega\dot{u}\sin\lambda + \frac{\tilde{g}}{l}u = 0.$$

Let us put $u(t) = U(t)e^{irt}$, substitute in the previous equation and choose $r = -\Omega\sin\lambda$ in order to get rid of the $\dot{U}$ term. Defining

$$\omega^2 = \frac{g}{l} + \Omega^2\sin^2\lambda - \Omega^2\frac{R_E}{l}\cos^2\lambda.$$

the resulting equation is written as $\ddot{U} + \omega^2 U = 0$.

One has $\Omega^2 \approx 10^{-9}\,s^{-1}$ while $\Omega^2(R_E/l) \approx 10^{-3}\,s^{-1}$. It is thus fully justified to neglect the second term as compared to the third one so that:

$$r = -\Omega\,\sin\lambda;$$
$$\omega^2 = \omega_0^2 - \Omega^2\frac{R_E}{l}\cos^2\lambda,$$

where $\omega_0 = \sqrt{g/l}$ is the proper angular frequency of the pendulum in a Galilean frame. The solution of the equation $\ddot{U} + \omega^2 U = 0$ is trivial and gives $U = X + iY = Ae^{i\omega t} + Be^{-i\omega t}$.

It is always possible to choose the origin of time and the axis orientation in order to obtain $X(t) = A\cos\omega t$; $Y(t) = B\sin\omega t$. In this system of reference, the pendulum describes an ellipse with an angular frequency ω. In the complex plane, the multiplication by e^{irt} to switch from the set X, Y to the set x, y is just a rotation of angle rt. In other words, the axes of the ellipse turn slowly in time with the angular velocity $|r| = \Omega\sin\lambda$.

At the equator $\lambda = 0$ so that $r = 0$ and

$$\omega = \sqrt{\omega_0^2 - \Omega^2\frac{R_E}{l}}.$$

The pendulum oscillates with an angular frequency slightly smaller than its proper value.

At the pole $\lambda = \pi/2$, then $r = -\Omega$ and $\omega = \omega_0$. The pendulum oscillates with its proper angular frequency and the ellipse axes make a complete revolution in one day (see Fig. 2.7).

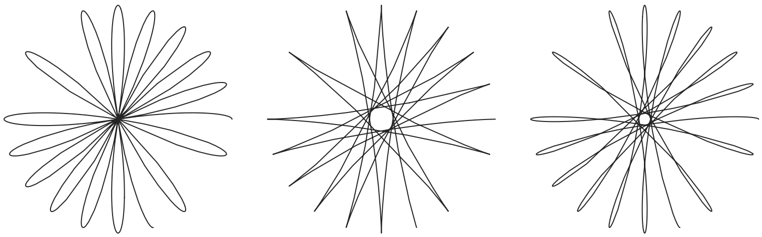


Fig. 2.7 Different types of trajectories for the ellipse drawn on the ground for three different initial release conditions. For these three cases $r/\omega = 1/10$. On the left hand side, the pendulum is released with a tangential velocity opposite to the driving velocity; in the middle, the pendulum is released with no initial velocity and on the right hand side one has a situation intermediate between the previous cases

~~2.5. Three-particle System [Statement p. 61]~~

~~A – Changing coordinates~~

~~1. In changing the origin $q'_i = q_i - a$, the velocities do not vary $\dot{q}'_i = \dot{q}_i$, neither do the relative distances $q'_i - q'_j = q_i - q_j$. The Lagrangian is invariant and one deduces the following constant of the motion (this is also a consequence of Noether's theorem):~~

$$\cancel{P = \sum_i \partial_{\dot{q}_i} L,}$$