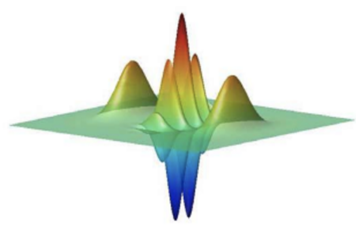
The quantum phase space formalism

How to introduce a phase space picture for quantum mechanics.



1 Problem

For a Hamiltonian system, the phase space trajectories $(\mathbf{x}(t), \mathbf{p}(t))$ uniquely determine the evolution of any dynamical system. Each point $(\mathbf{x}_0, \mathbf{p}_0)$ in phase space belong to one trajectory, and as time progresses the state of the corresponding system follows the trajectory. Without a doubt, picturing dynamics in phase space is a valuable tool for building a deeper intuition for physical systems. Early on (1930'th) people started to think if there is an analogue in quantum mechanics. It is clear that there cannot be a direct translation to quantum mechanics due to the Hesineberg uncertainty relation $\Delta x \Delta p \geq \hbar/2$. As a result, we must consider distributions, obeying the uncertainty relation, instead of single points as in classical mechanics. Hence, we cannot have trajectories in a quantum mechanical phase space formalism. Similarly, we could also imagine probability distributions $P(\mathbf{x}, \mathbf{p}, t)$ in the classical case (what defines a classical probability distribution?). How to construct the quantum distributions is not unique, but can be done in many different ways. There are three main distributions: *Husimi*'s $Q(\alpha)$-function, *Glauber*'s $P(\alpha)$-function, and the *Wigner* function $W(x, p)$.

Let us limit the analysis to a single particle on a line, *i.e.* a one dimensional problem with a two dimensional phase space. Discuss how one finds/defines the phase space distributions. In which sense are they proper probability distributions? How are they used for calculations (*i.e.* how do they replace the wave-function $\psi(x, t)$)? The wave-functions $\psi(x, t)$ are derived from the Schrödinger equation, but which equation do the distributions obey?

References

[1] Chapter 3: M. O. Scully and M. S. Zubairy, *Quantum Optics*, (Cambridge University Press, 1997).