Problems, *Analytical mechanics* FK7049

Suggested problems for first tutorial:

Generalized coordinates, d'Alembert's principle, variational problems, the Lagrangian

1 Newtonian mechanics I

For a set of point particles with masses m_i and momentum $\mathbf{p}_i$, express $\dot{\mathbf{p}}_i$ in terms of the forces and furthermore what is the total angular momentum $\mathbf{L}$? Show that the $\mathbf{L}$ is preserved whenever the particles are not exposed to any external forces. If $\mathbf{R}$ is the center-of-mass position with respect to some reference system, and $\mathbf{P}$ the center-of-mass momentum, show that the total momentum

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} + \sum_i \mathbf{r}_i' \times \mathbf{p}_i',$$

where $\mathbf{r}_i'$ is the position of particle i with respect to the center-of-mass, and for the total kinetic and total potential energies

$$T = \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i v_i'^2,$$

$$V = \sum_i V_i + \frac{1}{2} \sum_{i \neq j} V_{ij}.$$

Is the total energy conserved?

2 Newtonian mechanics II

You are provided an infinitely strong rope, and your task is to "hang it up". You stand on the equator and let the rope point straight out from the equator, such that one end of the rope touches the ground. How long is the rope such that it hangs freely?

3 Generalized coordinates I

In the lecture we solved the problem of the Atwood's machine using the Lagrange equations. The problem comprises two masses m_1 and m_2 (see Fig. 1.7 in Goldstein) and thereby we have 2×3 degrees of freedom. However, the constrains reduces this to only a single degree of freedom. Find these constrains.

4 Generalized coordinates II

A mass point m rolls frictionlessly on the inner surface of a circular cone (cone angle α) in the gravitational field of the earth, see Fig. 1. Formulate the constraints and choose appropriate generalized coordinates.

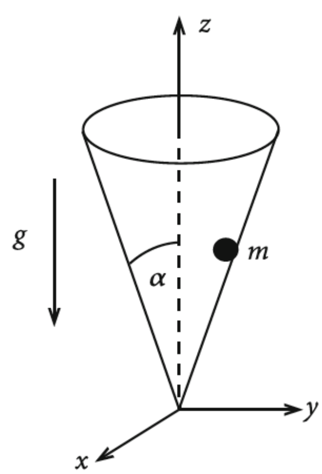

Figure 1: Setup of problem 4.

5 d'Alembert I

Two material points (particles) P_1 and P_2, of masses m_1 and m_2, are connected by a massless, inextensible, perfectly malleable wire of length l. Neglecting friction and supposing that the two material points can move on the semicircle $x^2 + y^2 = R^2$, $z = 0$, $y > 0$ as in the Fig. 2, determine the equilibrium position of the system by means of the D'Alembert's principle of virtual work, if $2l > \pi R$.

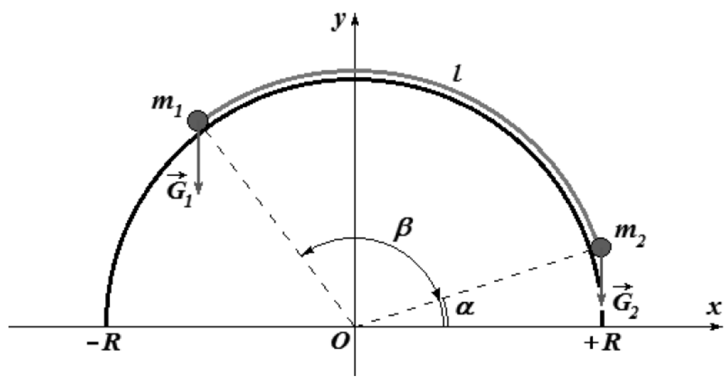

Figure 2: Setup of problem 5.

6 Generalized potentials

The Lorentz force for a charge particle (charge $\tilde{q}$) in an electromagnetic field, expressed in terms of the scalar and vector potential, reads

$$\mathbf{F}(\mathbf{r}, t) = \tilde{q}\left[-\nabla\varphi(\mathbf{r}, t) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t) + (\mathbf{v} \times (\nabla \times \mathbf{A}(\mathbf{r}, t)))\right]. \qquad (1)$$

With the generalized potential

$$U(\mathbf{r}, \dot{\mathbf{r}}, t) = \tilde{q}\left[\varphi(\mathbf{r}, t) - \mathbf{v} \cdot \mathbf{A}(\mathbf{r}, t)\right],$$

show that the generalized force

$$Q_j = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}$$

reproduces the correct Lorentz force (1).

7 Lagrange I

A mass m rotates frictionlessly on a tabletop. Via a thread of the length l ($l = r + s$) it is connected through a hole in the table with another mass M, see Fig. 3. How does M move under the influence of the gravitational force?

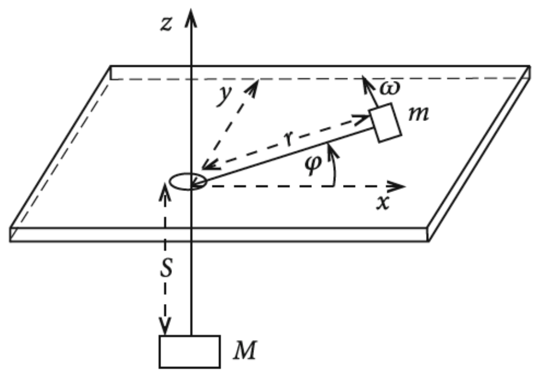

Figure 3: Setup of problem 7.

a) Formulate and classify the constraints!

b) Find the Lagrangian and its equations of motion!

c) Under what conditions does the mass M slip upwards or downwards?

d) Discuss the special case $\omega = 0$.

8 Lagrange II

If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations, show by direct substitution that

$$L' = L + \frac{d}{dt}F(q_1, q_2, ..., q_n, t)$$

also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments. This is called a *mechanical gauge transformation.*

(Problem 8 in chapter 1 of Goldstein.)

9 Lagrange III

The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\mathbf{A} \to \mathbf{A} + \nabla\psi(\mathbf{r}, t),$$

$$\phi \to \phi - \frac{1}{c}\frac{\partial\psi}{\partial t},$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a charged particle moving in the electromagnetic field? Is the motion affected?

(Problem 8 in chapter 1 of Goldstein.)

10 Lagrange IV

Let $q_1, ..., q_n$ be a set of independent generalized coordinates for a system of n degrees of freedom, with a Lagrangian $L(q, \dot{q}, t)$. Suppose we transform to another set of independent coordinates $s_1, ..., s_n$ by means of transformation equations

$$q_i = q_i(s_1, ..., s_n), \qquad i = 1, ..., n.$$

Such a transformation is called a *point transformation*. Show that if the Lagrangian function is expressed as a function of s_i, $\dot{s}_j$, and t through the equations of transformation, then L satisfies Lagrange's equations with respect to the s coordinates:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_j}\right) - \frac{\partial L}{\partial s_j} = 0.$$

In other words, the form of Lagrange's equations is invariant under a point transformation.

(Problem 10 in chapter 1 of Goldstein.)

11 Lagrange V – *Double pendulum*

The planar double pendulum is pictured in Fig. 4; two pendula, of masses m_1 and m_2, and lengths l_1 and l_2 (the bars are assumed massless and stiff), are connected according to the figure.

a) Formulate and classify the constraints.

b) Find the Lagrangian and its equations of motion.

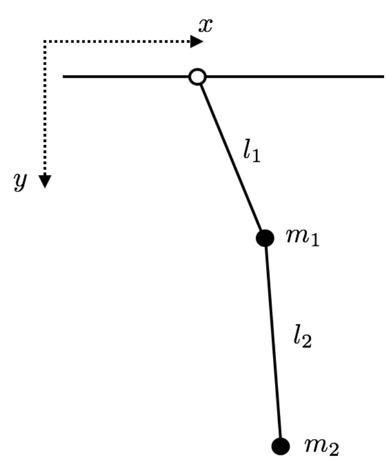

Figure 4: Double pendulum problem 11.

12 Lagrange VI

Let us consider a pendulum of length l, and where one mass m_1 moves without friction along a horizontal line and the other mass m_2 is exposed to the gravitational field, see Fig. 5.

a) Formulate and classify the constraints!

b) Find the Lagrangian and its equations of motion!

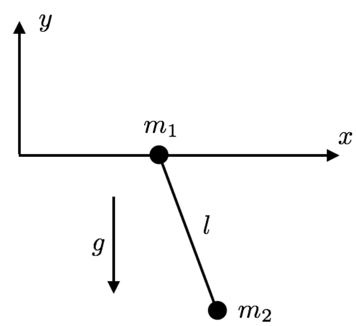

Figure 5: Double pendulum problem 12.

13 Lagrangian VII

Consider again a bead of mass m frictionlessly gliding on a wire which rotates with constant angular velocity ω and the bead shall now additionally move in the a gravitational field, see Fig. 6.

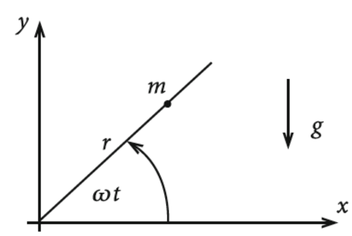

Figure 6: Setup of the rotating bead problem in the presence of gravity 13.

a) Which constraint forces are present?

b) Formulate the Lagrangian function for the bead.

c) Determine the Lagrange equation of motion and find its general solution.

d) Use the initial conditions

$$r(t=0) = r_0, \qquad \dot{r}(t=0) = 0.$$

How large must ω be at the least to force the bead to move outwards for $t \to \infty$?

e) How would we have to treat the problem in Newton?s mechanics?

14 Lagrange VIII – Cylindical coordinates

Let the position of a particle be described by cylindrical coordinates (ρ, φ, z). The potential energy of the particle is given as

$$V(\rho) = V_0 \ln\left(\frac{\rho}{\rho_0}\right),$$

with V_0 and ρ_0 two constants.

a) Write down the Lagrangian.

b) Formulate the Lagrange equations of motion.

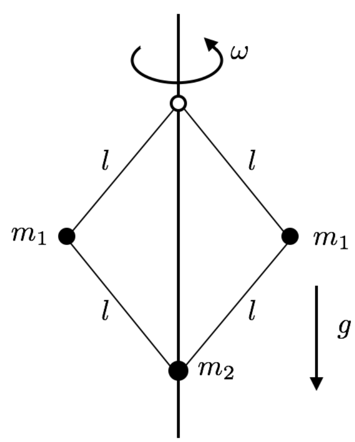

Figure 7: Setup of problem 15.

15 Lagrange IX

Two equal masses m_1 are connected to a point (circle in figure) by two identical massless threads of length l. A third mass m_2 is connected with the two other masses by yet two more identical threads, see Fig. 7. The whole setup rotates around the center axis with a constant frequency ω, and are exposed to a gravity field. Find the generalized coordinates and write down the Lagrangian.

16 Lagrange X

Consider the Lagrangian

$$L = \frac{1}{2}\sum_{i,j} M_{ij}(q)\dot{q}_i\dot{q}_j,$$

where the matrix elements of M_{ij} depend on the configuration space coordinates, and the matrix is assumed to have an inverse M_{ij}^{-1}. Write down the Lagrange's equations and solve for the accelerations.

17 Lagrange XI

Consider the system in fig. 8, consisting of two identical springs and two identical masses. The spring constants are k and their natural lengths a, and the mass is m. The masses can only move vertically.

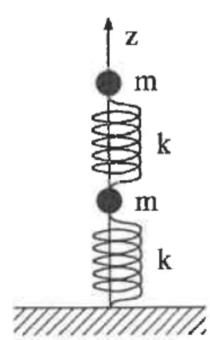

Figure 8: Setup of problem 17.

a) Write down the kinetic energy, potential energy and the Lagrangian for the system.

b) Find the equilibrium positions z_1 and z_2.

c) Write down the Lagrange's equations.

d) By solving the Lagrange's equation, find the angular frequency for the system.

Suggested problems for second tutorial:
Variational methods, Legendre transforms, and Hamilton's principle

18 Functionals

Let $f(x;\lambda)$ be a one-parameter set of functions defined by

$$f(x;\lambda) = \cos(x+\lambda) \tag{2}$$

and let's consider the functional F defined by

$$F[f(x)] = \int_{\pi/2}^{\pi/2} \mathrm{d}x f(x). \tag{3}$$

Extremize F with respect to the one-parameter set of functions $f(x;\lambda)$, i.e. find λ such that $F[f(x;\lambda)]$ takes extremal values. Are they maxima, minima, saddle points?

19 Variational I – *Isoperimetric problem*

Determine the form of a plane curve of a given length l which encloses a surface of maximum area.

(*Hint*: Use Lagrange multipliers to take the constraint into account.)

20 Variational II – *Catenary problem*

Determine the curve formed by a rope or chain of uniform density and perfect flexibility, hanging freely between two points of suspension, not in the same vertical line. The rope has a fixed length (it is not extensible).

21 Variational III – *Triathlon*

See Fig. 9: A triathlete starts at a point A on a beach, a distance l_1 away from the waterfront. A buoy, a distance l_2 from the shore, is out in the water at point B. The two points, A and B, are not vertical to each other, but horizontally apart by l_3. On the beach the triathlete runs with a velocity v_1 and in the water she swims with a velocity v_2. By optimizing the time to get to the buoy, with what angle, relative the horizontal shore line, should the triathlete hit the water?

Which physical law did you just derive?

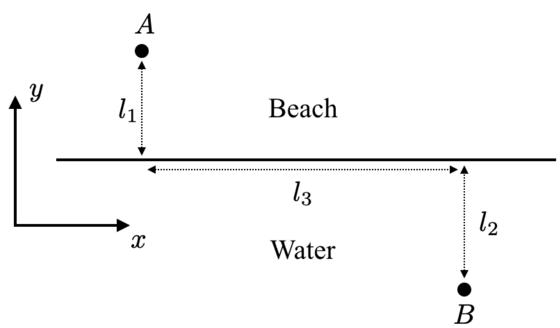

Figure 9: Triathlon problem.

22 Variational IV – *Cylinder*

Use the Euler-Lagrange equations to find the shortest distance between two points p_1 and p_2 on a cylinder, see Fig. 10.

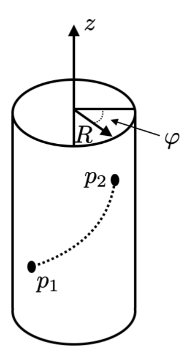

Figure 10: Shortest distance between two points, p_1 and p_2, on a cylinder.

23 Legendre Transform

For α and β constants:

a) Determine the Legendre transform $g(u)$ of the function $f(x) = \alpha x^2$.

b) Determine the Legendre transform $g(x, u)$ of the function $f(x, y) = \alpha x^2 y^3$.

c) Determine the Legendre transform $g(u)$ of the function $f(x) = \alpha(x + \beta)^2$.

d) Determine the Legendre transform $g(x, u)$ of the function $f(x, y) = \alpha x^3 y^5$.

(For checking your results you may perform the inverse transform!)

24 Hamiltonian in different coordinates

Write the Hamiltonian of a particle of mass m, situated in a conservative force field, in Cartesian, cylindrical, polar, spherical, parabolic, and elliptic coordinates.

Suggested problems for 3'rd tutorial:
Hamiltons equations, symmetries,

25 Hamilton's principle

For a free particle, consider the action integral

$$S = \frac{m}{2}\int_{t_1}^{t_2} \dot{x}^2 dt.$$

Evaluate this integral for an $x(t)$ that solves the equations of motion, and express the answer as a function of t_1, t_2, and the initial and final positionsx_1 and x_2. Repeat the exercise for a harmonic oscillator.

26 Hamilton's equations I

Using the Hamiltonian formalism, write the differential equations-of-motion of a particle of mass m, sliding without friction on a fixed sphere of radius R (*spherical pendulum* - see Fig. 11).

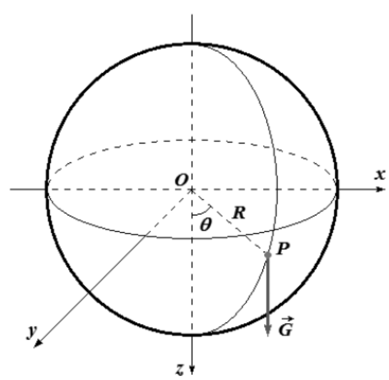

Figure 11: Spherical pendulum problem 26.

27 Hamilton's equations II

Let $H(q, p)$ be a Hamiltonian and $(q(t), p(t))$ a corresponding trajectory in phase space, with energy E. We consider now a new Hamilton function

$K(q,p)$, which is an arbitrary function of the original Hamiltonian $K(q,p) = F(H(q,p))$. We denote $(\tilde{q}(t), \tilde{p}(t))$ the corresponding trajectories with energy $F(E)$. Show that these trajectories are the same as the previous ones, but governed by a different temporal law.

28 Lagrange equations from Hamiltons equations

Given the Hamilton function $H = H(\mathbf{q}, \mathbf{p}, t)$ of a mechanical system and its equations of motion). The Lagrangian is the negative Legendre transform of the Hamilton function

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \sum_{j=1}^{N} p_j \frac{\partial H}{\partial p_j} - H.$$

Use Hamilton?s equations of motion in order to derive with this relation the Lagrange equations of motion of the second kind.

29 Symmetries I

A particle of mass m performs a two-dimensional motion in the xy-plane under the influence of the force

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}(\mathbf{x}, \mathbf{y}) = -\left(\alpha + \frac{\beta}{r}\right)\mathbf{r},$$

where α and β are positive constants. Choose plane polar coordinates (r, φ) as generalized coordinates.

a) Write down the kinetic and the potential energy in plane polar coordinates.

b) Calculate the generalized momenta p_r and p_φ.

c) Formulate the Hamilton function! Find and interpret two integrals of motion.

30 Symmetries II

The potential energy of a particle of mass m is given in cylindrical coordinates (ρ, φ, z) by

$$V(\rho) = V_0 \ln \frac{\rho}{\rho_0},$$

with the constants V_0 and ρ_0.

a) What is the Hamilton function?

b) Derive Hamilton?s equations of motion.

c) Find three conservation laws.

31 Symmetries III

In a two-dimensional space, a particle of mass m, located by its Cartesian coordinates x,y, is subject to a potential of the form $V(x - 2y)$.

a) Write down the Lagrangian of the system. If x increases by the arbitrary quantity s, what is the increase in y in order for the potential to be unchanged? Deduce that the Lagrangian is invariant under such translations.

b) Show that the quantity $\dot{x} + \frac{1}{2}\dot{y}$ is a constant of the motion.

32 Central force potential I

A particle of mass m moves in three dimensions under a central force

$$F(r) = -\frac{dV(r)}{dr},$$

Find the Lagrangian expressed in terms of the spherical coordinates (r, θ, ϕ), and derive the corresponding Lagrange equations. Using the Lagrange equations, show that $mr^2\dot{\phi}\sin^2(\theta)$ is a constant of the motion.

33 Central force potential II - spherical pendulum

An inextensible string of length l is fixed at one end, and has a bob of mass m attached at the other. The bob swings freely in three dimensions under gravity, and the string remains taut, so the system is a spherical pendulum. Find the Lagrangian in an appropriate coordinate system, and identify a conserved quantity. Write down the Lagrange equations.

34 Phase space I – Pendulum

Consider the simple planar pendulum of length l and with mass m. Write down the Lagrangian, and from it derive the Hamiltonian. Find the Hamilton equations and their fixed points and discuss their stability. What kind of fixed points do you have? Sketch how the phase diagram looks. You're welcome to also solve the problem numerically and use the numerical solutions to plot the phase diagram. How would the phase diagram modify when we take friction into account?

Suggested problems for 4'th tutorial:
Canonical transformations,

35 Poisson brackets I

Determine the Poisson brackets which are built by the Cartesian components of the linear momentum $\mathbf{p}$ and the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of a particle with mass m.

36 Poisson brackets II

Prove the relation

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k,$$

where ε_{ijk} is the Levi-Chivita antisymmetric tensor, and the L_i's the angular momentum variables. Note that I use curly brackets $\{..,..\}$ for the Poisson brackets, and not $[..,..]$ as in Goldstein.

37 Poisson brackets III

Given the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ for a particle of mass m, calculate $\{\mathbf{L}^2, L_{x,y,z}\}$ where $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$. If two of the components of $\mathbf{L}$ are integrals of motion, show that also the third component is an integral of motion. (Assume that $\mathbf{L}$ is time-independent.).

38 Poisson brackets IV

A particle of mass m moves in a central field. Demonstrate by use of the Poisson bracket that the z-component L_z of the angular momentum is an integral of motion.

39 Poisson brackets V

Consider an infinitely differential function $A = A(\mathbf{q}(t), \mathbf{p}(t))$ with no explicit time-dependence, i.e. $\partial A/\partial t = 0$. Assume also no explicit time-dependence of the Hamiltonian, $\partial H/\partial t = 0$. Use a Taylor expansion of A to express the time-dependence of A in terms of H and $A(0) = A(\mathbf{q}(0), \mathbf{p}(0))$.

40 Poisson brackets VI

If you're up for it, prove the *Jacobi identity*

$$\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0.$$

Note that I use curly brackets $\{.., ..\}$ for the Poisson brackets, and not $[.., ..]$ as in Goldstein.

41 Poisson brackets VII

Prove *Poisson's theorem* that states that the Poisson bracket of two integrals of motion is again an integral of motion.

42 Poisson brackets VIII

Check whether for the linear harmonic oscillator the mechanical observable

$$f(q, p, t) = p \sin \omega t - m\omega q \cos \omega t$$

is an integral of motion. Validate the result by a direct calculation of df/dt.

43 Canonical transformations 0

Derive the identities of the second column of Table 9.1 in Goldstein, page 373.

44 Canonical transformations I

Is it possible that two components of the angular momentum, e.g. L_x and L_y, appear simultaneously as canonical momenta?

45 Canonical transformations II

Show that the transformation defined by

$$Q = \ln\left(\frac{\sin p}{q}\right), \qquad P = q \cot p$$

is canonical, given that (q, p) are canonical.

46 Canonical transformations III

The same as the previous problem, but for the transformation

$$Q = \ln\left(1 + \sqrt{q}\cos p\right), \qquad P = 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p.$$

Show also that the transfomration is generated by

$$F_3(p, Q) = -\left(e^Q - 1\right)^2 \tan p.$$

47 Canonical transformations IV

a) For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2},$$

show that there is a constant of the motion

$$D = \frac{pq}{2} - Ht.$$

b) As a generalization of part a), for motion in a plane with the Hamiltonian

$$H = |\mathbf{p}|^n - ar^{-n},$$

where $\mathbf{p}$ is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of the motion

$$D = \frac{\mathbf{p} \cdot \mathbf{r}}{n} - Ht.$$

c) The transformation $Q = \lambda q$ and $p = \lambda P$ is obviously canonical. However, the same transformation with t time dilatation, $Q = \lambda q$, $p = \lambda P$ and $t' = \lambda^2 t$ is not. Show that, however, the equations of motion for q and p for the Hamiltonian in part a) are invariant under the transformation. The constant of the motion D is said to be associated with this invariance.

48 Canonical transformations V

Prove directly that the transformation

$$Q_1 = q_1, \qquad P_1 = p_1 - 2p_2,$$

$$Q_2 = p_2, \qquad P_2 = -2q_1 - q_2$$

is canonical and find a generating function.

49 Canonical transformations VI

By any method you choose, show that the following transformation is canonical:

$$x = \frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right), \qquad p_x = \frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right),$$

$$y = \frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right), \qquad p_y = -\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right),$$

where α is some fixed parameter.

Apply this transformation to the problem of a particle of charge q moving in a plane that is perpendicular to a constant magnetic field $\mathbf{B}$. Express the Hamiltonian for this problem in the $(Q_i,\, P_i)$ coordinates, letting the parameter α take the form

$$\alpha = \frac{qB}{c}.$$

From this Hamiltonian obtain the motion of the particle as a function of time.

50 Canonical transformations VII

A mechanical system with the Hamilton function

$$H = \frac{1}{2m} p^2 q^4 + \frac{k}{2q^2}$$

given, as well as the generating function of a canonical transformation

$$F_1(q, Q) = -\sqrt{mk}\frac{Q}{q}.$$

a) What are the transformation formulas

$$p = p(Q, P), \qquad q = q(Q, P)?$$

b) What is the Hamiltonian function $K = K(Q, P)$?

c) Find the solutions of the problem for the variables Q, P?

51 Canonical transformations VIII

For which values α and β is the transformation

$$Q = q^\alpha \cos(\beta p), \qquad P = q^\alpha \sin(\beta p)$$

canonical?

Suggested problems for 5'th tutorial:
Hamilton-Jacobi theory, Rigid body motion

52 Hamilton-Jacobi theory I

For the initial conditions $p(0) = 0$ and $q(0) = q_0$ (meaning that the particle starts at its turning point) we solved the Harmonic oscillator from the Hamilton-Jacobi equation in the lecture. In particular, with the given initial conditions the solutions read

$$q(t) = \sqrt{\frac{2E}{m\omega^2}} \cos(\omega t), \qquad p(t) = -\sqrt{2Em} \sin(\omega t).$$

While deriving this result we found the generating function on the form

$$S(q, \alpha, t) = m\omega \int dq \sqrt{\frac{2\alpha}{m\omega^2} - q^2} - \alpha t.$$

Use the above to show that the generating function S is indeed the action integral

$$S = \int L\, dt + C,$$

where L is the Lagrangian and C a constant of integration.

53 Hamilton-Jacobi theory II

Find the Hamilton-Jacobi differential equation for the one-dimensional movement of a particle of mass m in the potential

$$V(q) = -bq$$

and solve the problem with the initial condition

$$q(0) = q_0, \qquad p(0) = p_0.$$

54 Hamilton-Jacobi theory III

A particle of mass m moves in one dimension in the potential

$$V(q) = c\,\mathrm{e}^{\gamma q}, \qquad c,\, \gamma \in \mathbb{R}.$$

Find $q(t)$ and $p(t)$ using the Hamiltoni-Jacobi method.

55 Hamilton-Jacobi theory IV

A charged particle is constrained to move in a plane under the influence of a central force potential (non-electromagnetic) $V(r) = \frac{1}{2}kr^2$, and a constant magnetic field $\mathbf{B}$ perpendicular to the plane, so that

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}.$$

Set up the Hamilton-Jacobi equation for Hamilton's characteristic function in plane polar coordinates Separate the equation and reduce it to quadratures. Discuss the motion if the canonical momentum p_θ is zero at $t = 0$.

56 Hamilton-Jacobi theory V

Suppose the potential in a problem of one degree-of-freedom is linearly dependent on time, such that the Hamiltonian has the form

$$H = \frac{p^2}{2m} - mAtx,$$

where A is a constant. Solve the dynamical problem by means of Hamilton's principle function, under the initial condition $t = 0$, $x(0) = 0$, and $p(0) = mv_0$.

57 Action-angle variables I

A particle moves in in periodic motion in one dimension under the influence of a potential $V(x) = F|x|$, where F is a constant. Using action-angle variables, find the period of the motion as a function of the particles energy.

58 Action-angle variables II

A particle of mass m is constrained to move in the vertical plane defined by the parametric equations

$$x = l(2\phi + \sin 2\phi), \qquad y = l(1 - \cos\phi).$$

There is the usual constant gravitational force acting in the vertical y direction. By the method of action-angle variables find the frequency of oscillations for all initial conditions such that the maximum of ϕ is less than or equal to $\pi/4$.

59 Action-angle variables III

Apply the method of the action-angle variables to determine the frequencies of a three-dimensional harmonic oscillator with pairwise different force constants.

60 Central potential I – Diatomic molecule

We consider vibrations (*i.e.* neglect center-of-mass translations and rotations) of a diatomic molecule where the two nuclei have masses m_1 and m_2 and they interact by the *Morse potential*

$$V(\mathbf{x}_1 - \mathbf{x}_2) = V_0 \left(1 - e^{-a(|\mathbf{x}_1 - \mathbf{x}_2| - r_e)}\right)^2, \tag{4}$$

which is shown in Fig. 12. Here, V_0 is the potential strength/amplitude, $a > 0$ determines the range of the potential, and r_e sets the minimum position.

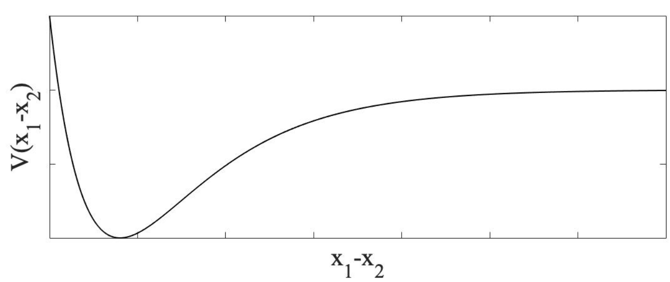

Figure 12: Morse potential describing vibrations in a diatomic molecule of problem 60.

a) Find the generalized coordinates and write down the Lagrangian.

b) Apply the Legendre transform to get the Hamiltonian, and write down the corresponding Hamilton's equations of motion.

c) Sketch the phase space!

61 Central potential II – Yukawa potential

Same as the previous problem but for the *Yukawa potential*

$$V(r) = -g^2 \frac{e^{-kmr}}{r},$$

where g is the potential amplitude, m the mass of the particle, and k a (positive) scaling constant.

62 Steiner's theorem

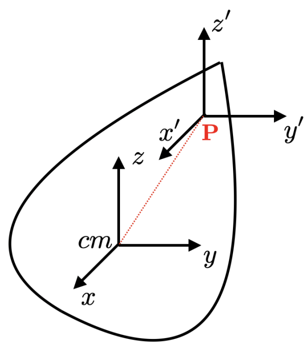

Figure 13: Reference frames of the rigid body of problem 62.

Let $\mathbf{I}$ be the inertia tensor for a rigid body calculated according to the center-of-mass body-fixed system $\bar{K}$. Further let $\bar{K}'$ be a different body-fixed system given by translation of $\bar{K}$ by a vector $\mathbf{P}$, see Fig. 13. Prove that the inertia tensor $\mathbf{I}'$ calculated according to the second system is

$$I'_{\mu\nu} = I_{\mu\nu} + M\left(P^2\delta_{\mu\nu} - P_\mu P_\nu\right). \tag{5}$$

63 Rigid body

Consider the pyramid of fig. 14, with a mass m, hight h and quadratic base with side a. Find the center-of-mass position, and furthermore, chose a

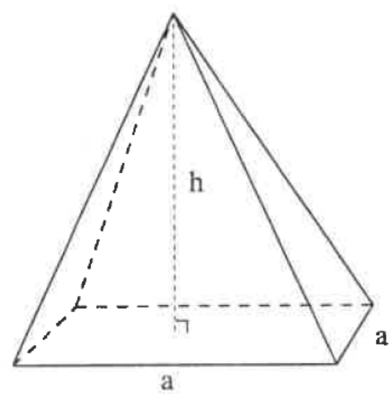

Figure 14: Pyramid for problem 63.

center-of-mass coordinate system and calculate the inertia tensor.

64 Inertia tensor I

a) A rigid body has density $\rho(\mathbf{r})$, show that

$$\mathbf{I} = \int d^3x \rho(\mathbf{r}) \begin{bmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_2 x_1 & x_3^2 + x_1^2 & -x_2 x_3 \\ -x_3 x_1 & -x_3 x_1 & x_1^2 + x_2^2 \end{bmatrix}$$

b) Prove that the inertia tensor is symmetric.

c) Assume we have two rigid bodies a and b at a fixed distance from one another. We construct a common reference system for both bodies (for example their common center-of-mass) and calculate their respective inertia tensors $\mathbf{I}_a$ and $\mathbf{I}_b$. Show that the total inertia tensor for the two bodies $\mathbf{I}_{tot} = \mathbf{I}_a + \mathbf{I}_b$, *i.e.* the inertia tensor is *additative.*

d) Let I_i, $i = 1, 2, 3$ be the eigenvalues of some inertia tensor, show that the inertia tensor is positive, *i.e.*

$$I_i \geq 0, \quad i = 1,\, 2,\, 3,$$

and further that

$$I_1 + I_2 \geq I_3, \quad I_2 + I_3 \geq I_1, \quad I_1 + I_3 \geq I_2.$$

e) Assume a rigid body has a symmetry plane, *i.e.* the body is symmetric with respect to reflection in the plane. Show that the center-of-mass lies in the plane, and so does to of the bodies principle axes, while the third principle axis is perpendicular to the plane.

65 Inertia tensor II

Calculate the inertia tensor in the center-of-mass reference system for

a) A homogeneous ball of mass M and radius R.

b) A hallow ball of mass M and radius R.

c) A spherical symmetric ball of mass M, density $\rho(r) \propto r^n$ (n an integer) and radius R. Which values of n are allowed?

d) A homogeneous cylinder of mass M, radius R and hight h.

e) A hallow cylinder of mass M, radius R and hight h.

f) A homogenous cuboid of mass M and sides a, b, and c.

g) A 2D equilateral triangle of mass M and sides a.

h) A 2D equilateral triangle with three equal masses $M/3$ in each corner and connected with massless rods of length a.

66 Inertia tensor III

Repeat the same as the previous problem, but now for a reference system at the edge of the rigid body (note that "the edge" is not uniquely defined for all bodies).

67 Moments of inertia I

The inertia tensor of a rigid body is found to have the form

$$\mathbf{I} = \begin{bmatrix} I_{11} & I_{12} & 0 \\ I_{21} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}, \qquad I_{12} = I_{21}.$$

Determine the three moments of inertia and consider the following special cases:

a) $I_{11} = I_{22} = A$, $I_{12} = B$. Can I_{33} be arbitrary?

b) $I_{11} = A$, $I_{22} = 4A$, $I_{12} = 2A$. What can you say about I_{33}? What is the shape of the body in this example?

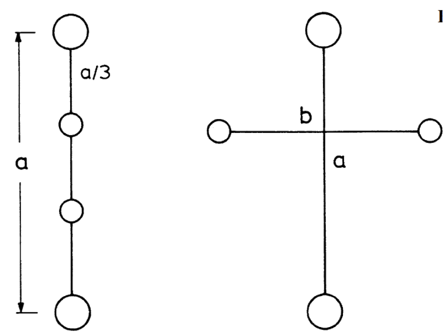

Figure 15: Configurations of masses for problem 68

68 Moments of inertia II

Calculate the moment of inertia I_3 for two arrangements of four balls, two heavy (radius R, mass M) and two light (radius r, mass m) with homogeneous mass density, as shown in Fig 15.

69 Moments of inertia III

Calculate the moments of inertia of a torus filled homogeneously with mass. Its main radius is R; the radius of its section is r.

70 Rigid body motion I

A homogenous cone of mass m, height h and base radius r is rolling on a plane without slipping, see fig. 16. The angular velocity is ω. Calculate the kinetic energy.

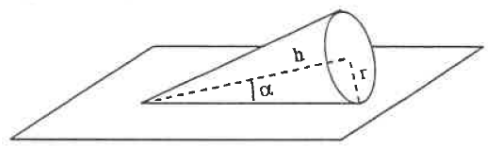

Figure 16: Rolling cone for problem 70

71 Rigid body motion II

Return to the Atwood machine of Fig. 17; two masses m_1 and m_2 connect by a massless string of length l and under influence of gravity. The string is over a pulley of radius r and moments of inertia I, and we can neglect any friction. Find the generalized coordinates and write down the Lagrangian. Write down the Lagrange's equation and solve it for suitable initial conditions.

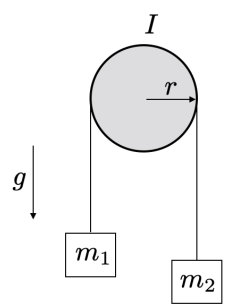

Figure 17: The Atwood machine of problem 71 with a pulley with moment of inertia I.

72 Rigid body motion III

On the inner surface of a cylinder side wall (radius R) rolls another cylinder (radius r, mass density ρ =const), see Fig. 18. write down the Lagrangian and derive the Lagrange's equations. For small deflections φ, solve the equations of motion.

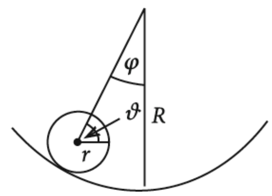

Figure 18: Rolling cylinder of problem 72.

73 Rigid body motion IV

We consider the planar pendulum with a massless rod and a homogenous circular disc with mass m attached to the end of the rod, see Fig. 19. The length of the rod is R and the radius of the disc r, and furthermore, the disc can rotate without any friction around its center. Write down the Lagrangian and the Lagrange's equations for the system. Like for the double pendulum we have two generalized coordinates, argue if this implies that the system becomes chaotic (like the double pendulum) or not. How would things change if the disc experiences some friction as it rotates around its center?

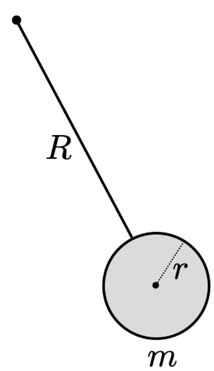

Figure 19: Circular disc pendelum of problem 73.

74 Rigid body motion V

The same problem as 73, but now we assume that the disc is fixed, i.e. unable to rotate.