Lösningar

Tentamen i Analytisk Mekanik den 16 mars 2018, under tiden 08.00-13.00. Lärare: Ingemar Bengtsson. Hjälpmedel: Penna, suddgummi och linjal. Bedömning: 4 poäng/uppgift. Betyg: 0-3 = F, 4-6 = Fx, 7-9 = E, 10-12 = D, 13-16 = C, 16-20 = B. Kompletterande muntlig tentamen efter överenskommelse.

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1. For what values of the constants a, b, c, d is the transformation

$$q \to Q = aq + bp \ , \quad p \to P = cq + dp \tag{1}$$

canonical?

2. Consider three equal masses rigidly fixed to sit at the corners of an equilateral triangle. Compute the inertia tensor with respect to the centre of mass, and with respect to the position of one of the masses.

3. Consider the Euler–Lagrange equations coming from the Lagrangian

$$L = \frac{1}{2} g_{ij} \dot{x}_i \dot{x}_j \ , \tag{2}$$

where it is understood that the matrix elements of g_{ij} may depend on the coordinates x_i. Show that they can be written in the form

$$\ddot{x}_i = \frac{1}{2} g^{-1}_{im} (\partial_m g_{jk} - \partial_j g_{mk} - \partial_k g_{mk}) \dot{x}_j \dot{x}_k \ . \tag{3}$$

Warning: There is a deliberate misprint in the formula, but that should be easy to correct.

4. The virial theorem states that for bounded motion in homogeneous potentials the time averages of the kinetic and potential energies obey

$$2\langle T \rangle = \beta \langle V \rangle \ , \tag{4}$$

where β is the degree of homogeneity of the potential. Compute β for a cluster of stars bound together by gravity. Show that the theorem implies

that the stars will (on average) move slower if energy is injected into the cluster.

5. The Hamiltonian for a particle in free fall is

$$H = H(q, p) = \frac{p^2}{2m} + gmq \ . \tag{5}$$

Find a canonical transformation to new variables Q and P such that

$$H(q(Q, P), p(Q, P)) = P \ . \tag{6}$$

Solve Hamilton's equations in terms of Q and P, transform that solution back to the variables q and p, and check that you have the correct general solution of the original problem.

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1. Let $Q = aq + bp$, $P = cq + dp$

Then

$$\{Q,P\} = \{aq+bp, cq+dp\} = /\text{linearity}/ =$$

$$= ad\{q,p\} + bc\{p,q\} = /\text{anti-symmetry}/ =$$

$$= ad - bc$$

Hence the transformation is canonical iff

$$ad - bc = 1$$

Remark: Write $\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$

Then $ad-bc=1$ means that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant 1.

2. 

Place the origin at the c.o.m., and the masses at

$$R\begin{pmatrix} \sqrt{3}/2 \\ -1/2 \\ 0 \end{pmatrix}, \quad R\begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \\ 0 \end{pmatrix}, \quad R\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The inertia tensor with respect to the origin is

$$I_{ij} = mR^2\left\{\delta_{ij} - \begin{pmatrix} 3/4 & -\sqrt{3}/4 & \\ -\sqrt{3}/4 & 1/4 & \\ & & 0 \end{pmatrix} + \delta_{ij} - \begin{pmatrix} 3/4 & \sqrt{3}/4 & \\ \sqrt{3}/4 & 1/4 & \\ & & 0 \end{pmatrix} + \delta_{ij} - \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}\right\} =$$

$$= mR^2\begin{pmatrix} 3-3/2 & & \\ & 3-1/2-1 & \\ & & 3 \end{pmatrix} = \frac{3}{2}mR^2\begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$$

To get it with respect to the point $\begin{pmatrix} 0 \\ R \\ 0 \end{pmatrix}$, use Steiner's theorem (and the total mass $3m$):

$$I'_{ij} = I_{ij} + 3mR^2\left(\delta_{ij} - \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}\right) = I_{ij} + 3mR^2\begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} =$$

$$= \frac{3m}{2}R^2\begin{pmatrix} 3 & & \\ & 1 & \\ & & 4 \end{pmatrix}$$

Check mentally that everything makes sense!

3. The E-L equations are

$$\frac{d}{dt}\left(g_{ij}\dot{x}_j\right) = \frac{1}{2}\partial_i g_{jk}\dot{x}_j\dot{x}_k$$

(Note the handling of indices on RHS.)

$$\Rightarrow \quad g_{ij}\ddot{x}_j + \dot{g}_{ij}\dot{x}_j =$$

$$= g_{ij}\ddot{x}_j + \partial_k g_{ij}\dot{x}_k\dot{x}_j = \frac{1}{2}\partial_i g_{jk}\dot{x}_j\dot{x}_k$$

Let the matrix g_{ij} have the inverse $g^{-1}{}_{ij}$,

$$g^{-1}{}_{ik}g_{kj} = g_{ik}g^{-1}{}_{kj} = \delta_{ij}$$

Then

$$\ddot{x}_i = g^{-1}{}_{im}\left(\frac{1}{2}\partial_m g_{jk}\dot{x}_j\dot{x}_k - \partial_k g_{mj}\dot{x}_k\dot{x}_j\right)$$

This can be written more elegantly!

$$\ddot{x}_i = g^{-1}{}_{im}\left(\frac{1}{2}\partial_m g_{jk} - \partial_k g_{mj}\right)\dot{x}_j\dot{x}_k =$$

$$= \frac{1}{2}g^{-1}{}_{im}\left(\partial_m g_{jk} - \partial_j g_{mk} - \partial_k g_{mj}\right)\dot{x}_j\dot{x}_k$$

4. For a cluster of stars the potential is a sum of terms like

$$\frac{G m_1 m_2}{|\underline{x}_1 - \underline{x}_2|}$$

Every term has degree of homogeneity -1.

Hence

$$\langle 2T + V\rangle = 0$$

The energy is $E = T + V$, so

$$\langle E\rangle = \langle T + V\rangle = -\langle T\rangle,$$

by the virial theorem. Since $T \geqslant 0$, the bound cluster has $E < 0$. If energy is injected, the absolute value of E shrinks, and so does $\langle T\rangle$. The stars are moving slower on average!

5. Clearly we must have $P = \frac{p^2}{2m} + gmq$.

A canonical transformation is completed by

$$Q = -\frac{1}{gm} p \quad \Rightarrow \quad \{Q, P\} = 1$$

This transformation is easily inverted,

$$p = -gm\, Q$$

$$q = \frac{1}{gm}\left(P - \frac{p^2}{2m}\right) = \frac{1}{gm}\left(P - \frac{g^2 m}{2} Q^2\right)$$

In the new variables Hamilton's eqs. are

$$\dot{Q} = \frac{\partial H}{\partial P} = 1 \qquad\qquad Q = t + c_1$$

$$\Rightarrow$$

$$\dot{P} = -\frac{\partial H}{\partial Q} = 0 \qquad\qquad P = c_2$$

where c_1 and c_2 are constants. It follows that

$$q(t) = q(Q(t), P(t)) = \frac{1}{gm}\left(c_2 - \frac{g^2 m}{2}(t + c_1)^2\right)$$

Renaming a constant this is the familiar

$$q(t) = k_1 - \frac{g}{2}(t + c_1)^2$$

Comments on solutions:

- One student had the brilliant idea of placing the three masses at $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$, and then using Steiner twice. The catch (if the problem had been properly formulated) is that you then have to diagonalize to get the moments of inertia.

- Almost everyone was a bit vague about what the potential actually looks like in problem 4. This would have been worth a comment, given that there may be 10^5 stars.