Tentamen i Analytisk Mekanik den 20 mars 2015, under tiden 9.00-14.00. Lärare: Ingemar Bengtsson. Hjälpmedel: Penna, suddgummi och linjal. Bedömning: Uppgift 4 poäng/uppgift. Betyg: 0-2 = F, 3-6 = Fx, 7-9 = E, 10-12 = D, 13-16 = C, 17-20 = B. Grades are raised one step if you have the bonus points.

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*If there is a clear sky, I will come around 10.30, and students who wish to do so may then follow me out (in a group) to watch the Solar Eclipse for not more than half an hour.*

**1.** Assuming $\Omega^2 \neq k/m$, find the general solution of the equation

$$m\ddot{x} + kx = f\cos(\Omega t) \ . \tag{1}$$

**2.** The two-body central force problem can be reduced to that of motion in the effective potential

$$V_{\text{eff}} = \frac{l^2}{2mr^2} + V(r) \ . \tag{2}$$

Suppose that the potential $V(r) = -kr^{\beta}$. Investigate for what values of $k$ and $\beta$ you can have stable circular motion.

**3.** The inertia tensor is defined by

$$I_{ij} = \sum m(x^2\delta_{ij} - x_i x_j) \ , \tag{3}$$

where the sum is over all individual masses making up a rigid body. Write down and prove the inequalities obeyed by its eigenvalues.

**4.** Consider the 2-particle Lagrangian

$$L = \frac{m_1}{2}\dot{x}_i^{(1)}\dot{x}_i^{(1)} + \frac{m_2}{2}\dot{x}_i^{(2)}\dot{x}_i^{(2)} - V(|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|) \ . \tag{4}$$

Show that it transforms into a total derivative under the transformation



$$\delta x_i^{(1)} = \delta x_i^{(2)} = -v_i t \ , \tag{5}$$

where $v_i$ is a constant and "infinitesimal" but otherwise arbitrary vector. Use Noether's theorem to construct the corresponding conserved charge $Q_i$. Physically, what does this symmetry mean?

**5.** Let $Q_i$ be the conserved charge from the previous exercise. Express it as a function on phase space. Calculate

$$\delta x_i^{(1)} = \{x_i^{(1)}, v_j Q_j\} \quad \text{and} \quad \dot{Q}_i = \{Q_i, H\} + \frac{\partial Q_i}{\partial t} \ , \tag{6}$$

where $H = H(x,p)$ is the Hamiltonian of the system and $v_i$ is a constant but otherwise arbitrary vector.

# ANALYTISK MEKANIK 20 mars - 15, lösningar

1. It is convenient to rewrite the equation as

$$\ddot{x} + \omega^2 x = \frac{f}{m}\cos(\Omega t)$$

where $\omega^2 = k/m$ is assumed positive.

The general solution is the sum of any solution + the general solution of the homogeneous equation (with $f=0$). The latter is

$$x_{hom}(t) = A\cos(\omega t + \delta)$$

and contains two arbitrary constants, an amplitude and a phase.

To find a particular solution we make a guess, say

$$x_{part}(t) = B\cos(\Omega t)$$

where $B$ is a constant to be determined.

Inserting our Ansatz we find

$$B(-\Omega^2 + \omega^2)\cos\Omega t = \frac{f}{m}\cos\Omega t$$

with the solution

$$B = \frac{f}{m(\omega^2 - \Omega^2)}$$

$\therefore$

The general solution of the problem is

$$x(t) = \frac{f}{m(\omega^2-\Omega^2)}\cos(\Omega t) + A\cos(\omega t + \delta)$$

It contains two arbitrary constants, as it should.

2. A circular orbit has $r =$ constant.

A stable circular orbit exists if there is an $r$ such that

$$V'_{eff}(r) = 0$$

$$V''_{eff}(r) > 0$$

(Making rough sketches of what

$$V_{eff}(r) = \frac{L^2}{2mr^2} - k r^\beta$$

looks like for various choices of $\beta$ is instructive at this point. Note that we have to assume $L^2 \neq 0$. )

Anyway

$$V'_{eff} = -\frac{L^2}{\cancel{2}mr^3} - \beta k r^{\beta-1} = -\frac{1}{r^3}\left(\frac{L^2}{m} + \beta k r^{\beta+2}\right)$$

$$V''_{eff} = \frac{3L^2}{mr^4} - \beta(\beta-1) k r^{\beta-2}$$

A solution for $V'_{eff} = 0$ exists only if $\boxed{\beta k < 0}$ ($m$ is positive!), and is

$$r = \left(-\frac{L^2}{m\beta k}\right)^{\frac{1}{\beta+2}}$$

Concerning stability, at this value of $r$

$$V''_{eff} = \frac{1}{r^4}\left(\frac{3L^2}{m} - \beta(\beta-1) k r^{\beta+2}\right) =$$

$$= \frac{1}{r^4}\left(\frac{3L^2}{m} + \beta(\beta-1) k \frac{L^2}{m\beta k}\right) =$$

$$= \frac{L^2}{mr^4}\left(3 + \beta - 1\right) =$$

$$= (\text{something positive}) \times (\beta + 2)$$

So stability requires $\beta > -2$.

$\therefore$

The conditions on $k$ and $\beta$ are

$$\beta k < 0$$
$$\beta > -2$$

or

$$-2 < \beta < 0 \quad \text{if} \quad k > 0$$
$$\beta > 0 \quad \text{if} \quad k < 0$$

In particular, $\beta = -1$ and $k > 0$ (Newton) and $\beta = 2$ and $k < 0$ (Hooke) are OK.

3. The inertia tensor is a symmetric matrix, and we know that there exists a coordinate system (principal axes) in which it is diagonal.

We apply the definition in this coordinate system! Then

$$I = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix} = \begin{pmatrix} \sum_{\text{part}} m(y^2+z^2) & & \\ & \sum m(x^2+z^2) & \\ & & \sum m(x^2+y^2) \end{pmatrix}$$

with zeroes off the diagonal. By inspection we see that

$$I_1 \geq 0 \qquad I_2 \geq 0 \qquad I_3 \geq 0$$

Also

$$I_1 + I_2 = \sum m(x^2+y^2+2z^2) \geq \sum_m m(x^2+y^2) = I_3$$

Similarly

$$I_2 + I_3 \geq I_1$$
$$I_3 + I_1 \geq I_2$$

4. The physical meaning of the statement is that if we have a solution, and change the velocities of the particles with the same constant amount, we obtain another solution.

To see if this is true, we apply the transformation. Computing to first order in $V_i$

$$\delta L = m_1 \dot{x}_i^{(1)} (\delta \dot{x}_i^{(1)}) + m_2 \dot{x}_i^{(2)} (\delta \dot{x}_i^{(2)}) + \delta V =$$

/ noting that $\delta V = 0$ due to the special form of $V(\underline{x}^{(1)}, \underline{x}^{(2)})$ / =

$$= -m_1 \dot{x}_i^{(1)} \frac{d}{dt}(V_i t) - m_2 \dot{x}_i^{(2)} \frac{d}{dt}(V_i t) =$$

$$= -V_i (m_1 \dot{x}_i^{(1)} + m_2 \dot{x}_i^{(2)}) = \frac{d}{dt}(\Lambda)$$

where we defined

$$\Lambda = -V_i (m_1 x_i^{(1)} + m_2 x_i^{(2)})$$

In this situation Noether's theorem says that there exists a conserved charge $Q_i$,

$$V_i Q_i = \delta x_i^{(1)} \frac{\partial L}{\partial \dot{x}_i^{(1)}} + \delta x_i^{(2)} \frac{\partial L}{\partial \dot{x}_i^{(2)}} - \Lambda =$$

$$= -V_i t (m_1 \dot{x}_i^{(1)} + m_2 \dot{x}_i^{(2)}) + V_i (m_1 x_i^{(1)} + m_2 x_i^{(2)}) =$$

$$= V_i \left[ m_1 x_i^{(1)} + m_2 x_i^{(2)} - t m_1 \dot{x}_i^{(1)} - t m_2 \dot{x}_i^{(2)} \right]$$

It is convenient to introduce the centre of mass

$$X_i = \frac{1}{m_1 + m_2} (m_1 x_i^{(1)} + m_2 x_i^{(2)})$$

Redefining $Q_i$ with a factor $(m_1 + m_2)$ we see that the statement is that

$$Q_i(t) = X_i(t) - t \dot{X}(t)$$

is constant – which is the statement that the centre of mass does not accelerate.

5. The canonical momenta are

$$p_i^{(1)} = \frac{\partial L}{\partial \dot{x}_i^{(1)}} = m\dot{x}_i^{(1)} \quad , \quad p_i^{(2)} = m\dot{x}_i^{(2)}$$

The Hamiltonian is

$$H = \frac{1}{2m_1} p_i^{(1)} p_i^{(1)} + \frac{1}{2m_2} p_i^{(2)} p_i^{(2)} + V(|\underline{x}^{(1)} - \underline{x}^{(2)}|)$$

and the conserved charges are

$$Q_i = m_1 x_i^{(1)} + m_2 x_i^{(2)} - t p_i^{(1)} - t p_i^{(2)}$$

Now, using canonical Poisson brackets

$$\{x_i^{(1)}, v_j Q_j\} = \{x_i^{(1)}, -t v_i p_i^{(1)}\} = -t v_i$$

and

$$\dot{Q}_i = \{m_1 x_i^{(1)} + m_2 x_i^{(2)} - t p_i^{(1)} - t p_i^{(2)}, H\} - p_i^{(1)} - p_i^{(2)} =$$

$$= m_1 \{x_i^{(1)}, \frac{1}{2m_1} p_i^{(1)} p_i^{(1)}\} + m_2 \{x_i^{(2)}, \frac{1}{2m_2} p_i^{(2)} p_i^{(2)}\} -$$

$$- t\{p_i^{(1)}, V(|\underline{x}^{(1)} - \underline{x}^{(2)}|)\} - t\{p_i^{(2)}, V(x^{(1)}, x^{(2)})\} -$$

$$- p_i^{(1)} - p_i^{(2)} =$$

$$= p_i^{(1)} + p_i^{(2)} + t\frac{\partial V}{\partial x_i^{(1)}} + t\frac{\partial V}{\partial x_i^{(2)}} - p_i^{(1)} - p_i^{(2)} =$$

$$= 0 \qquad (\text{eftersom } \partial_{x_i^{(1)}} V = -\partial_{x_i^{(2)}} V)$$