Toroidal Compactification In Bosonic String Theory

So far, we have only discussed bosonic strings in non-compact spaces, where, X^μ is a periodic function of σ:

$$X^\mu(\sigma+2\pi) = X^\mu(\sigma)$$

As a result, in the mode expansions of X^μ_L and X^μ_R given by

$$X^\mu_R = \frac{1}{2}x^\mu_R + \frac{\alpha'}{2}p^\mu_R(\tau-\sigma) + \text{oscillators}$$

$$X^\mu_L = \frac{1}{2}x^\mu_L + \frac{\alpha'}{2}p^\mu_L(\tau+\sigma) + \text{oscillators},$$

we have $\quad p^\mu_L = p^\mu_R$.

Let us now consider the case when one direction, say the 25^{th} direction, is a circle of radius R. This allows close strings to have windings around this circle. The periodicity conditions are then modified to

$$X^{25}(\sigma+2\pi) = X^{25}(\sigma) + 2\pi n R \qquad (22)$$

$$X^\mu(\sigma+2\pi) = X^\mu(\sigma), \text{ for } \mu \neq 25.$$

This means that the closed strings winds n times ~~around~~ around the compact dimension of space. For the coordinate $X^{25} = X^{25}_L + X^{25}_R$, we get:

$$X^{25} = X_L^{25} + X_R^{25} = x^{25} + \frac{\alpha'}{2}\left(p_L^{25} + p_R^{25}\right)\tau + \frac{\alpha'}{2}\left(p_L^{25} - p_R^{25}\right)\sigma + \text{osci...}$$

Momentum Modes: As before, $p_L^{25} + p_R^{25} = 2p^{25}$, where p^{25} is the center of mass momentum of the string along X^{25}. As far as this motion is concerned, we can regard the string as a point particle moving on a circle of radius R. The momentum is, therefore, quantized as

$$p^{25} = \frac{m}{R} \qquad (m = \text{integer})$$

Winding Modes: The boundary condition on X^{25} implies that

$$\frac{\alpha'}{2}\left(p_L^{25} - p_R^{25}\right) = nR \qquad (n = \text{integer})$$

Thus, we get:

$$\boxed{X^{25} = x^{25} + \underbrace{\alpha' \frac{m}{R}\tau}_{\text{momentum mode}} + \underbrace{nR\sigma}_{\text{winding mode}} + \text{oscillators}} \qquad (23)$$

$$\left.\begin{aligned} p_L^{25} - p_R^{25} &= \frac{2nR}{\alpha'} \\ p_L^{25} + p_R^{25} &= 2\frac{m}{R} \end{aligned}\right\} \Longrightarrow \boxed{\begin{aligned} p_L^{25} &= \frac{nR}{\alpha'} + \frac{m}{R} \\ p_R^{25} &= \frac{m}{R} - \frac{nR}{\alpha'} \end{aligned}} \qquad (24)$$

Mass Spectrum

$$L_0 = \frac{\alpha'}{4} p_R^\mu p_{R\mu} + \sum_{n>0} \alpha_{-n}\cdot\alpha_n \qquad (\mu = 0,..,25)$$

$$= \frac{\alpha'}{4} p_R^{25} p_R^{25} + \frac{\alpha'}{4} p^i p_i + \sum_{n>0} \alpha_{-n}\cdot\alpha_n, \quad (i = 0,..,24)$$

$$\bar{L}_0 = \frac{\alpha'}{4} p_L^\mu p_{L\mu} + \sum_{n>0} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$$

$$= \frac{\alpha'}{4} p_L^{25} p_L^{25} + \frac{\alpha'}{4} p^i p_i + \sum_{n>0} \tilde{\alpha}_{-n}\tilde{\alpha}_n$$

where, $p_L^i = p_R^i = p^i$ for $i = 0, .. 24$, which spans the physical part of the space. The mass is identified as

$$M^2 = -p^i p_i \qquad \text{(this is the mass in 25 dimensions and not in 26 dimensions)}$$

$\Rightarrow$

$$L_0 = \frac{\alpha'}{4} p_R^{25} p_R^{25} - \frac{\alpha'}{4} M^2 + N_R$$

$$\bar{L}_0 = \frac{\alpha'}{4} p_L^{25} p_L^{25} - \frac{\alpha'}{4} M^2 + N_L.$$

Translation invariance along σ implies that for physical states,

$$(L_0 - \bar{L}_0)|phys\rangle = 0$$

$$(25) \quad \Rightarrow \quad \boxed{\frac{\alpha'}{4} p_R^2 + N_R = \frac{\alpha'}{4} p_L^2 + N_L} \quad \text{(for allowed states)}$$

(compare with the non compact case)

As in the non compact bosonic case, the mass-shell conditions are:

$$(L_0 - 1)|phys\rangle = 0 \Rightarrow \alpha' M^2 = \alpha'(P_R^{25})^2 + 4(N_R - 1)$$

$$(\bar{L}_0 - 1)|phys\rangle = 0 \Rightarrow \alpha' M^2 = \alpha'(P_L^{25})^2 + 4(N_L - 1).$$

Eqn. (25) guarantees the consistency of these two expressions for M^2. Using (24) and writing M^2 in a symmetric way, we get:

$$\boxed{\alpha' M^2 = \alpha'\left(\frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2}\right) + 2N_R + 2N_L - 4} \quad (26)$$

Note that for non-zero winding, $P_L^{25} \neq P_R^{25}$ and, therefore, $N_L \neq N_R$. For states of zero winding, $N_L = N_R$ as in the non-compact case.

Condition on N_L and N_R

Let us now look at eqn. (25) more carefully. From (24) it follows that:

$$N_R - N_L = \frac{\alpha'}{4}\left[(P_L^{25})^2 - (P_R^{25})^2\right] = nm$$

$$\boxed{N_R - N_L = nm} \quad (27)$$

m: momentum quantum number.

n: winding quantum number.

Using this, the mass formula can be written in an alternative form:

$$\boxed{\alpha' M^2 = \alpha'\left(\frac{m}{R} - \frac{nR}{\alpha'}\right)^2 + 4N_R - 4} \qquad (28)$$

Remarks

1) The mass formula in superstring theory, with one direction compactified, has essentially the same form as in (26) or (28). In this case, N_L and N_R also ~~...~~ contains fermionic oscillators coming from worldsheet fermions. A second difference is that the constant "4" is replaced by other numbers. This is because in $(L_0 - a)|phys\rangle = 0$, the normal ordering constant "a" is no longer $a=1$.

2) ~~...~~ For curiosity, let's look at states with $N_R = 0$. Then the mass formula (28) is simplified to

$$\boxed{\alpha' M^2_{N_R=0} = \alpha'\left(\frac{m}{R} - \frac{nR}{\alpha'}\right)^2 - 4}$$

Of course, for these states, N_L is not necessarily zero and is determined by (27). These states do not have any particular significance in bosonic string theory. However, the analogous states in superstring theory saturate the BPS bound and, thus, belong to short representations of the space-time supersymmetry algebra. They play an important role in recent developments.

T-Duality in Bosonic String Theory

Consider the mass formula (26):

$$\alpha' M^2 = \alpha'\left(\frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2}\right) + 2N_R + 2N_L - 4 \qquad (26)$$

This was obtained by compactifying the 25^{th} dimension on a circle of radius R.

Note that ~~...~~ this can also be regarded as the mass formula for another theory in which the 25^{th} direction is compactified on a radius $\tilde{R}$, with

$$\tilde{R} = \frac{\alpha'}{R} \qquad (29)$$

To see this rewrite $\alpha' M^2$ in terms of $\tilde{R}$:

$$\alpha' M^2 = \alpha'\left(\frac{n^2}{\tilde{R}^2} + \frac{m^2 \tilde{R}^2}{\alpha'^2}\right) + 2N_R + 2N_L - 4$$

This has the same form as (26) with R replaced by $\tilde{R}$ and "m" and "n" interchanged.

This means that for any state (m, n) in a theory with radius R, one ~~can~~ finds a state (n, m) in a theory with radius $\tilde{R} = \alpha'/R$ such that the two states have the same mass:

$$M^2(m, n, R) = M^2(n, m, \tfrac{\alpha'}{R})$$

It turns out that the correspondence between theories with radii R and α'/R goes beyond the mass spectrum: In fact, the entire theory with radius R can be mapped to the theory with radius α'/R. This ~~real~~ relationship between the two theories is called T-duality.

<u>T-duality Map</u>: Here, we describe the map that relates the two theories: We constructed

$$X^{25} = X_L^{25} + X_R^{25} \quad , \quad \partial_- X_L^{25} = 0, \quad \partial_+ X_R^{25} = 0$$

~~We~~ Construct a new $\tilde{X}^{25}$ given by

$$\tilde{X}^{25} = \tilde{X}_L^{25} + \tilde{X}_R^{25}$$

where,

$$\boxed{\begin{aligned} \tilde{X}_L^{25} &= X_L^{25} \\ \tilde{X}_R^{25} &= -X_R^{25} \end{aligned}} \qquad (30)$$

Eqn. (30) is the T-duality map (when the theory is in flat-backgrounds). In terms of the full X^{25} it can be written as:

$$\boxed{\begin{aligned} \partial_+ \tilde{X}^{25} &= \partial_+ X^{25} \\ \partial_- \tilde{X}^{25} &= -\partial_- X^{25} \end{aligned}} \qquad (31)$$

It is easy to see that this map corresponds to the exchange $n \leftrightarrow m$, $R \leftrightarrow \alpha'/R$: ~~[illegible]~~

(24) $\Rightarrow$ $P_R = \frac{m}{R} - \frac{nR}{\alpha'}$, $\tilde{P}_R = \frac{\tilde{m}}{\tilde{R}} - \frac{\tilde{n}\tilde{R}}{\alpha'}$

(30) $\Rightarrow$ $\tilde{P}_R = -P_R = \frac{nR}{\alpha'} - \frac{m}{R}$

comparing the two expressions for $\tilde{P}_R$, we get:

$$\boxed{\tilde{m} = n, \quad \tilde{n} = m, \quad \tilde{R} = \frac{\alpha'}{R}}$$

The T-duality map (30) also maps the oscillators:

$$\tilde{\bar{\alpha}}^{25}_n = \bar{\alpha}^{25}_n \qquad \left(\begin{array}{l} \sim \;:\; \text{T-dual.} \\ - \;:\; \text{left mover} \end{array} \right).$$

$$\tilde{\alpha}^{25}_n = -\alpha^{25}_n$$

Under this, both N_L and N_R remain unchanged.

Remark : 1) ~~[illegible]~~ $(m, n=0) \longrightarrow (\tilde{m}=0, \tilde{n}=m)$

2) The T-duality map can be obtained in many ways: gauging a sigma model in different ways, canonical transformation etc. These are specially useful in the presence of background fields

3) We have discussed the action of T-duality in flat backgrounds. However, this transformation can also be formulated in the presence of

non-trivial $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. The condition is that G, B and Φ should not depend on X^{25}. The transformation $R \to \alpha'/R$ is then generalized to a transformation of the background fields:

4) The discussion of T-duality map here also applies to the superstring case, though in that case one should also extend the map to worldsheet fermions. We will see that in that case, T-duality maps II-A and II-B theories into each other.

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Transformation of Background Fields under T-duality

Let $X^{25} = \theta$, $X^{\mu} = X^{i}$ for $\mu = 0, \ldots 24$.

$$\tilde{G}_{ij} = G_{ij} - (G_{\theta i} G_{\theta j} - B_{\theta i} B_{\theta j})/G_{\theta\theta}$$

$$\tilde{B}_{ij} = B_{ij} + (B_{\theta i} G_{j\theta} - B_{\theta j} G_{i\theta})/G_{\theta\theta}$$

$$\tilde{G}_{\theta i} = B_{\theta i}/G_{\theta\theta}, \qquad \tilde{B}_{\theta i} = G_{\theta i}/G_{\theta\theta}$$

$$\tilde{G}_{\theta\theta} = 1/G_{\theta\theta},$$

$$\tilde{\phi} = \phi - \tfrac{1}{2}\log\left(-G_{\theta\theta}/-\tilde{G}_{\theta\theta}\right)$$

Thus string theory formulated around (G, B, ϕ) is equivalent to the one formulated around $(\tilde{G}, \tilde{B}, \tilde{\phi})$, even though from the field theory point of view the two backgrounds are very different from each other.

An Interesting Consequence of T-duality

consider a particle on a space where X^{25} is compactified on a circle of radius R. Then p^{25} is quantized in units of $\frac{1}{R}$. (there no winding no). As $R \to \infty$, p^{25} becomes a continuous variable and the theory has 26 dimensions. If $R \to 0$, then $p^{25} \to \infty$ and those modes are never excited. The theory is effectively 25 dimensional.

Now consider the string spectrum. We have $p^{25} = \frac{m}{R}$, $w = \frac{nR}{\alpha}$ and (with zero oscillator no)

$$M^2_{m,n} = p_m^2 + w_n^2 \qquad \text{discrete spectrum.}$$

i) As $R \to \infty$, $w_n \to \infty$ and p^{25} become continuous: 26 dim.
ii) As $R \to 0$, $p^{25} \to \infty$, but w_n is continuous: the theory is again 26-dimensional!

$\therefore$ It does not make sense to decrease R below the self-dual value of $R_0 = \frac{\alpha'}{R_0}$ or $R_0 = \sqrt{\alpha'}$. as we can always describe the theory in terms of variables where $R \geq R_0$.

Part II :

Dualities in Superstring Theory

In this section, I will first introduce type-IIA and type-II-B superstring theories, discussing the NS and R sectors and the GSO projection. Then, I will describe T-duality between type II-A and II-B theories, RR-fields and RR charged solitons, and S-duality in type II-B theory.

The Superstring Action and Supersymmetry Transformations :

The bosonic string theory has many drawbacks including:

i) The spectrum contains a tachyon.

ii) It does not have space-time supersymmetry, not even fermions.

Both these are overcome in a natural way, by introducing supersymmetry on the worldsheet. Below, we will see how this happens.

Consider the gauge fixed bosonic action (2) and add 'D' free fermions to it (D = dimension of space-time):

$$\boxed{S = -\frac{T}{2}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\,\bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu\right)} \quad (32)$$

Here, Ψ^μ: 2-component Majorana spinor on the world sheet.

ρ^α: Dirac matrices in 1+1 dimensions.

$\bar{\Psi}^\mu = \Psi^{\dagger\mu}\rho^0$

Note that while Ψ^μ is a spinor on the worldsheet, it is a vector in space-time. Besides the symmetries of the bosonic action, (32) is also invariant under supersymmetry transformations:

$$\delta X^\mu = \bar{\epsilon}\,\Psi^\mu$$

$$\delta \Psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon$$

The susy transformation parameter ϵ is a constant Majorana spinor.

Conventions: We choose the following conventions that makes the splitting between the left-moving and the right-moving sectors manifest:

coordinates: $\quad \sigma^\pm = \tau \pm \sigma \Rightarrow \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

or $\partial_\tau = \partial_+ + \partial_-$, $\partial_\sigma = \partial_+ - \partial_-$, $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Spinor Representation:

$$\Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}, \quad \bar{\Psi} = \Psi^\dagger \rho^0, \quad \{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$$

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho^3 = \rho^0\rho^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[We will see that $\partial_\pm \psi_\mp = 0$ which justifies the $\pm$ label on ψ]

With these conventions, (32) becomes:

$$\boxed{S = 2T\int d^2\sigma\, \partial_+ X^\mu \partial_- X_\mu + iT\int d^2\sigma \left(\psi_+^\mu \partial_- \psi_{+\mu} + \psi_-^\mu \partial_+ \psi_{-\mu}\right)}$$

(33)

Writing $\epsilon = \begin{pmatrix} \epsilon_- \\ \epsilon_+ \end{pmatrix}$, the susy transformations also breakup into independent left-moving and right-moving transformations:

ϵ_+ :

$$\left.\begin{aligned} \delta X^\mu &= i\epsilon_+ \psi_- \\ \delta \psi_-^\mu &= -2\partial_- X^\mu \epsilon_+ \end{aligned}\right\} \quad (\psi_-^\mu \leftrightarrow \partial_- X^\mu)$$

ϵ_- :

$$\left.\begin{aligned} \delta X^\mu &= -i\epsilon_- \psi_+^\mu \\ \delta \psi_+^\mu &= 2\partial_+ X^\mu \epsilon_- \end{aligned}\right\} \quad (\psi_+^\mu \leftrightarrow \partial_+ X^\mu)$$

Constraints:

Actions (32) or (33) are the gauge fixed forms of the superstring action. Before gauge fixing, the action has local $N=1$ susy ($\epsilon = \epsilon(\sigma,\tau)$). Thus, it also contains the superpartner of $h_{\alpha\beta}$ which is the 2-d gravitino.

This is eliminated by fixing the local $N=1$ susy. This leads to (32) which has $N=1$ superconformal symmetry. The constraints associated with gauge fixing are

$$\boxed{T_{\alpha\beta} = 0 \qquad J_\alpha^A = 0}$$

$T_{\alpha\beta}$: stress-energy tensor.
J_α^A: super current (A: spinor index).

These constraints fix the physical content of the theory.

Equations of Motion and Boundary Conditions:

The addition of free fermions modifies the worldsheet theory in a seemingly trivial way. However, it has drastic consequences for the space-time physics mainly due to the richer boundary conditions possible for the fermions. The bosonic part is the same as ~~before~~ the bosonic string theory, but now in $D=10$ ($\mu = 0, \ldots 9$).

Let us vary the fermionic action to derive the eqn. of motion:

$$S_F = iT\int d^2\sigma \left(\psi_+ \partial_- \psi_+ + \psi_- \partial_+ \psi_-\right)$$

$$\delta S_F = iT\int d^2\sigma \left[\delta\psi_+ \partial_- \psi_+ + \psi_+ \partial_-(\delta\psi_+) + (+\leftrightarrow -)\right]$$

$$= iT\int d^2\sigma \left[\delta\psi_+ \partial_- \psi_+ + \partial_-(\psi_+ \delta\psi_+) - \partial_-\psi_+ \delta\psi_+ + (+\leftrightarrow -)\right]$$

$$= iT\int d^2\sigma \Big[2\, \delta\psi_+ (\partial_- \psi_+) + 2\, \delta\psi_- (\partial_+ \psi_-) + \partial_-(\psi_+ \delta\psi_+) + \partial_+(\psi_- \delta\psi_-) \Big]$$

$\delta S_F = 0$ leads to the equations of motion

$$\boxed{\partial_+ \psi_- = 0 , \qquad \partial_- \psi_+ = 0} \tag{34}$$

provided

$$\int_0^{2\pi} d\sigma \int_{-\infty}^{+\infty} d\tau \left[\partial_+ (\psi_- \delta \psi_-) + \partial_- (\psi_+ \delta \psi_+) \right] = 0$$

As usual, we assume that

$$\delta \psi_\pm \Big|_{\tau = \pm\infty} = 0 ,$$

Then we should have

$$\int_0^{2\pi} d\sigma \left[\partial_\sigma (\psi_- \delta \psi_-) - \partial_\sigma (\psi_+ \delta \psi_+) \right] = 0$$

For a closed string, left-moving and right moving excitations are independent. Therefore, boundary terms for ψ_- and ψ_+ should vanish independently:

$$\psi_- \delta \psi_- \Big|_{\sigma = 2\pi} = \psi_- \delta \psi_- \Big|_{\sigma = 0}$$

$$\psi_+ \delta \psi_+ \Big|_{\sigma = 2\pi} = \psi_+ \delta \psi_+ \Big|_{\sigma = 0}$$

This is possible provided we restrict to a class of functions with the following boundary conditions:

$$\boxed{\psi_-^\mu (2\pi) = \pm \psi_-^\mu (0) , \qquad \psi_+^\mu (2\pi) = \pm \psi_+^\mu (0)} \tag{35}$$

Then the allowed variations $\delta\Psi_\pm$ have the same boundary conditions as $\Psi_\pm$ and the boundary term in δS_f vanishes.

$$\Psi_A^\mu(2\pi) = \Psi_A^\mu(0) \quad : \quad \text{Ramond (R) boundary condition}$$

$$\Psi_A^\mu(2\pi) = -\Psi_A^\mu(0) \quad : \quad \text{Neveu-Schwarz (NS) b.c.}$$

The boundary conditions can be chosen independently for the left and right-moving sectors. Therefore, the full Fock space contains four sectors: $|left\rangle \otimes |right\rangle$

NS−NS , NS−R, R−NS, RR.

Mode Expansion and Commutation Relations

The mode expansions and commutation relations for X^μ are the same as the bosonic case except for the fact that now $\mu = 0, \cdots 9$. Therefore, here we concentrate on the mode expansion for $\Psi_\pm^\mu$ in NS and R sectors. The procedure is the same for both left and right-movers.

NS - Sector:

$$\partial_\pm \Psi_\mp^\mu = 0 \Rightarrow \Psi_-^\mu = \Psi_-^\mu(\sigma^-), \quad \Psi_+^\mu = \Psi_+^\mu(\sigma^+).$$

Mode expansion:

$$\boxed{\begin{aligned} \Psi_-^\mu &= \sum_r b_r^\mu e^{-ir(\tau-\sigma)} \\ \Psi_+^\mu &= \sum_r \tilde{b}_r^\mu e^{-ir(\tau+\sigma)} \end{aligned}}$$

where, $r=$ integer $+ 1/2$. This half-odd integral moding insures that $\psi_\pm^\mu$ satisfy the NS b.c.

Quantization:

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s}, \quad \{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s}$$

$$\left([\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu} \right)$$

as before.

We can now construct the analogues of the number operators:

$$N^{(\alpha)} = \sum_{m\geq 1}^{\infty} \alpha_{-m}\cdot\alpha_m, \quad N^{(b)} = \sum_{r\geq 1/2}^{\infty} r\, b_{-r}\cdot b_r, \quad N_R = N^{(\alpha)} + N^{(b)}$$

(similarly for $\tilde{N}^{(\alpha)}$, $\tilde{N}^{(b)}$ and $N_L = \tilde{N}^{(\alpha)} + \tilde{N}^{(b)}$)

For $m>0$, $r>0$,

α_{-m}, b_{-r} : creation operators (they increase the eigenvalues of $N^{(\alpha)}$ and $N^{(b)}$ by "m" and "r" units respectively)

α_m, b_r : annihilation operators.

The Fock space vacuum (in the NS sector) is defined by

$$\alpha^\mu_n |0\rangle_{NS} = 0 \quad , \quad b^\mu_r |0\rangle_{NS} = 0 \qquad n, r > 0 .$$

Generic right moving state:

$$|N\rangle = \prod_j (\alpha_{-n_j})^{q_j} \prod_i (b_{-r_i})^{p_i} |0\rangle_{NS}$$

Then, $\quad N_R |N\rangle = \sum_j q_j n_j + \sum_i p_i r_i$

The same applies to the left-movers.

Note that from the space-time point of view, all these states are bosonic. From the world sheet point of view, states with odd number of fermionic excitations are ~~bosonic~~ fermionic, while those with even no. of fermionic excitations are bosonic. This is one of the ~~[illegible]~~ problems that will be resolved by the GSO projection to be discussed later.

R-Sector :

Space-time spinors have their origin in the R-sector. We will see how they emerge.

Mode Expansion :

$$\psi^\mu_- = \sum_m d^\mu_m \, e^{-in(\tau-\sigma)}$$

$$\psi^\mu_+ = \sum_m \tilde{d}^\mu_m \, e^{-in(\tau+\sigma)}$$

with this expansion, $\psi^\mu_\pm$ satisfy the R-b.c

Quantization:

$$\boxed{\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n}\,, \quad \{\tilde{d}^\mu_m, \tilde{d}^\nu_n\} = \eta^{\mu\nu}\delta_{m+n}}$$

The analogue of the number operators:

$$N^{(d)} = \sum_{m=1}^{\infty} m\, d_{-m}\cdot d_m\,, \qquad N_R = N^{(\alpha)} + N^{(d)}$$

$$N_L = \bar{N}^{(\alpha)} + \bar{N}^{(d)}.$$

For $m>0$,

d^μ_{-m} increases the $N^{(d)}$ eigenvalue by "m" units.

→ creation operator.

d^μ_m : annihilation operator

Fock space vacuum in the R-sector is defined by

$$\alpha^\mu_{+m}|0\rangle_R = 0\,, \qquad d^\mu_m|0\rangle_R = 0 \qquad (m>0).$$

Generic state

$$|N\rangle_R = \prod_j \prod_i (\alpha_{-n_j})^{q_j} (d_{-m_i})^{p_i} |0\rangle_R$$

The same applies to the left moving states.

Note that d^μ_0 does not appear in $N^{(d)}$ and to define $|0\rangle_R$, we have not yet specified the d^μ_0 action on it.

Emergence of Space-Time Spinors:

d_0^μ are neither creation nor annihilation operators as they commute with $N^{(d)}$. Thus, acting on any state, they generate a new state with the same $N^{(d)}$ eigenvalue. In particular $|0\rangle_R$ is a family of states, all satisfying $d_m^\mu |0\rangle_R = 0$ $(m>0)$, and related to each other by d_0^μ's. We will show that $|0\rangle_R$ is actually a 32 component spinor transforming under $SO(9,1)$:
The d_0^μ satisfy:

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \qquad (\text{same for } \tilde{d}_0^\mu)$$

which is essentially the clifford algebra in 9+1 dim. This can be made explicit by defining $\Gamma^\mu = i\sqrt{2}\, d_0^\mu$. Then

$$\boxed{\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}}$$

Since $[\Gamma^\mu, N^{(d)}]=0$, every state in the representation of the algebra for d_m^μ $(m\neq 0)$ should also form a ~~represent~~ representation of the algebra for Γ^μ, and therefore, correspond to a space-time spinor of $SO(9,1)$. In particular, this is the case with the Ramond vacuum $|0\rangle_R$.

To make this explicit, we use the notation $|0\rangle_R^a$ where, $a=1, 32$ is the spinor index. This is a

Majorana spinor(since $\psi_{\pm}^{\mu}$ are real).

We now give an explicit construction of $|0\rangle_R^a$:
[For simplicity, let us consider SO(10) rather than SO(9,1)]
Define operators a_k, $a_k^\dagger$ for $k=1,..,5$, such that

$$\Gamma_k = a_k + a_k^\dagger \quad ,(k=1,..5)$$

$$\Gamma_{5+k} = \frac{1}{i}(a_k - a_k^\dagger)$$

Then the Clifford algebra takes the form:

$$\{a_k, a_{k'}\} = 0 \quad , \quad \{a_k^\dagger, a_{k'}^\dagger\} = 0$$

$$\{a_k, a_{k'}^\dagger\} = \delta_{kk'} .$$

This is the usual algebra for 5 pairs of fermionic creation and annihilation operators. Define a ground state $|0\rangle$ annihilated by all a_k: $a_k|0\rangle = 0$. Then the full representation can be easily constructed as follows:

states	multiplicity		
$\|0\rangle$	1	= 1	
$a_i^\dagger\|0\rangle$	5	= 5	
$a_i^\dagger a_j^\dagger\|0\rangle$	$(5)(4)/2!$	= 10	
$a_i^\dagger a_j^\dagger a_k^\dagger\|0\rangle$	$(5)(4)(3)/3!$	= 10	$\underline{\underline{32}}$
$a_i^\dagger a_j^\dagger a_k^\dagger a_\ell^\dagger\|0\rangle$	$(5)(4)(3)(2)/4!$	= 5	
$a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger a_5^\dagger\|0\rangle$	1	= 1	

These are the 32 components of $|0\rangle_R^a$.

In 10-dimensions, the chirality operator is given by

$$\Gamma_{11} = \Gamma_0 \Gamma_1 \cdots \Gamma_9$$

It is possible to consider spinors of definite Γ_{11} chirality:

$$\Gamma_{11} \, |0\rangle_R^+ = +|0\rangle_R^+$$

$$\Gamma_{11} \, |0\rangle_R^- = -|0\rangle_R^-$$

$|0\rangle_R^+$ and $|0\rangle_R^-$ are 16 component Majorana-Weyl spinors. $|0\rangle_R$ can be decomposed as

$$\boxed{\underbrace{|0\rangle_R}_{32} = \underbrace{|0\rangle_R^+ \oplus |0\rangle_R^-}_{16 \oplus 16}}$$

The 16 components of $|0\rangle_R$ with even number of a_i's ~~bear~~ form a spinor with $\Gamma_{11} = +1$ (i.e., $|0\rangle_R^+$)

The 16 components with odd no. of a_i's form $|0\rangle_R^-$.

While both $|0\rangle_R^-$ and $|0\rangle_R^+$ are fermions in space-time, on the worldsheet, $|0\rangle_R^+$ is bosonic, while $|0\rangle_R^-$ is fermionic. It turns out that the physical spectrum contains either $|0\rangle_R^+$ or $|0\rangle_R^-$. This is acheived by the GSO projection.

The 32 component Ramond ground state in the left-moving part has the same structure as in the right moving part. However, the projection on $\Gamma_{11}=+1$ or $\Gamma_{11}=-1$ states can be made independently in the left and right-moving sectors. We come back to this in a while. One can now see that:

Fock space		space-time nature
$\lvert NS\rangle \otimes \lvert NS\rangle$	:	bosonic
$\lvert R\rangle \otimes \lvert R\rangle$	:	bosonic (bi spinor)
$\lvert R\rangle \otimes \lvert NS\rangle$	:	fermionic
$\lvert NS\rangle \otimes \lvert R\rangle$	:	fermionic

The Spin Field :

So far we have seen that the state $\lvert 0\rangle_R$ transforms as a space-time spinor under $SO(9,1)$. As we saw in the case of bosonic strings, states can usually be associated with operators. It is appealing to find an operator that ~~[illegible]~~ could create $\lvert 0\rangle_R$ from the NS vacuum $\lvert 0\rangle_{NS}$. This will unify the description of the NS and R sectors. If we denote this operator by S^a, then

$$\lvert 0\rangle_R^a \simeq S^a(0)\,\lvert 0\rangle_{NS}$$

Clearly, S^a should be a 32 component spinor field constructed out of the variables of the theory. Since $\lvert 0\rangle_R^a$ is fermionic, S^a is like a supersymmetry charge

Such an operator exists and is called the spin field. It can be constructed as follows:

Construction :

[For simplicity let us again consider $SO(10)$]

$|0\rangle_R^a$ was constructed as an $SO(10)$ spinor using the zero modes of ψ^μ. S^a is constructed using the full ψ^μ.

First, introduce 5 complex bosons ϕ^k $(k=1,..5)$ by bosonizing the 10 ψ^μ (this applies to both ψ_+^μ and ψ_-^μ):

$$\psi^1 \pm i\psi^2 = e^{\pm i\phi_1}, \quad \psi^3 \pm i\psi^4 = e^{\pm i\phi_2}, \quad \psi^5 \pm i\psi^6 = e^{\pm i\phi_3}$$

$$\psi^7 \pm i\psi^8 = e^{\pm i\phi_4}, \quad \psi^9 \pm i\psi^{10} = e^{\pm i\phi_5}.$$

In terms of ϕ_k introduced in this way, the spin operators are given by

$$e^{\pm i\phi_k/2} \qquad (h = \tfrac{1}{8},\ \bar{h} = 0)$$

Now consider the product

$$\boxed{S^{\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5\}} = e^{i\varepsilon_1\phi_1/2}\, e^{i\varepsilon_2\phi_2/2} \cdots e^{i\varepsilon_5\phi_5/2}}$$

where $\varepsilon_k = \pm 1$. Clearly for all possible combinations of ε_k, $S^{\{\varepsilon_k\}}$ has $(2)^5 = 32$ components. It can be shown that $S^{\{\varepsilon_k\}}$ is an $SO(10)$ spinor. This is the desired spin field and is essentially the vertex operator for the emission of space-time fermions. We use the notation $S^{\{\varepsilon_k\}} = S^a$, $a = 1,..32$.

S^a has conformal dimension $h = 5/8$. To get $h=1$, as for a legitimate vertex operator, the ghost field contribution must also be taken into account.

As for $|0\rangle_R^a$, S^a has ~~both~~ +ve and −ve chirality projections. The 16 component spinor obtained by taking even no. of +ve signs in $S^{\{\varepsilon_1, \varepsilon_2 \cdots \varepsilon_5\}}$ has $\Gamma_{11} = +1$. A similar construction applies to the left-movers.

So far, we have constructed the fock spaces for left and right moving oscillators, in both NS and R sectors of the theory. Now, we isolate the physical states by applying the constraints.

Applying the Constraints and the Mass Formula

classical constraints: $T_{\alpha\beta} = 0 \qquad J_\alpha^A = 0$.

NS - sector

$L_m\ (\bar{L}_m)$: Fourier modes of $T_{--}\ (T_{++})$

$G_r\ (\bar{G}_r)$: Fourier modes of $J_-\ (J_+)$ with NS bc.

The constraints are

$$\begin{cases} G_r |\phi\rangle = 0 & r > 0 \\ L_n |\phi\rangle = 0 & n > 0 \\ (L_0 - 1/2) |\phi\rangle = 0 & 1/2 : \text{ normal ordering constant.} \end{cases}$$

$$L_0 = \alpha'/4\, p_\mu p^\mu + N_R^{(\alpha)} + N_R^{(b)} \quad \Rightarrow$$

(NS).
$$\boxed{\alpha' M^2 = 4\left(N_R^{(\alpha)} + N_R^{(b)} - \tfrac{1}{2}\right)} \qquad (36)$$

We get an expression similar to (36) for the left movers. As usual, τ- translation invariance implies $(L_0 - \bar{L}_0)|phys\rangle = 0$.

<u>R- Sector</u> :

F_m $(\bar{F}_m)$: Fourier modes of J_- (J_+) with R-bc.

The constraints are

$$\begin{cases} F_n |f\rangle = 0 & m \geq 0 \\ L_m |f\rangle = 0 & m > 0 \\ L_0 |f\rangle = 0 & \text{normal ordering const.} = 0. \end{cases}$$

$$L_0 = \frac{\alpha'}{4} p_\mu p^\mu + N_R^{(\alpha)} + N_R^{(d)} \Rightarrow$$

$$(R): \quad \boxed{\alpha' M^2 = 4\left(N_R^{(\alpha)} + N_R^{(d)}\right)} \qquad (37)$$

with similar expressions for the left movers.

Let us now look at the states in the various sectors of the full Fock space $|left\rangle \otimes |right\rangle$.

<u>NS - NS Sector</u> :

The spectrum is determined by

$$\boxed{\alpha' M^2 = 4\left(N_R^{(\alpha)} + N_R^{(b)} - 1/2\right), \quad N_L^{(\alpha)} + N_L^{(b)} = N_R^{(\alpha)} + N_R^{(b)}} \qquad (38)$$

The low-lying states are.

$|0\rangle_{NS}$: $+\alpha' M^2 = -2$ (tachyon) $\quad [N^{(\alpha,b)}_{L,R} = 0]$

$b^{\mu}_{-1/2}\,\tilde{b}^{\nu}_{-1/2}|0\rangle_{NS}$: $M^2 = 0$ $(h_{\mu\nu}, b_{\mu\nu}, \phi)$ $\quad [N^{(\alpha)}_{L,R} = 0,\ N^{(b)}_{L,R} = 1/2]$

$\alpha^{\mu}_{-1}\,\tilde{\alpha}^{\nu}_{-1}|0\rangle_{NS}$: $\alpha' M^2 = 4$ (massive states) $\quad [N^{(\alpha)}_{L,R} = 1\ \ N^{(b)}_{L,R} = 0]$

states with $M^2 = 4/\alpha'$ has the same tensor structure as the massless ones (these are removed by GSO along with the tachyon).

R-R sector :

The spectrum is determined by

$$\boxed{\alpha' M^2 = 4(N^{(\alpha)}_R + N^{(d)}_R), \qquad N^{(\alpha)}_L + N^{(d)}_L = N^{(\alpha)}_R + N^{(d)}_R} \tag{39}$$

Massless states:

$|0\rangle^a_R \otimes |0\rangle^b_R$: $M^2 = 0$ $\left(N^{(\alpha),(d)}_{L,R} = 0\right)$

These bi-spinor states decompose into a collection of bosonic ~~fields~~ states, called the RR states. We will look at these states in more detail after introducing the GSO projection.

R-NS Sector :

As we expect, this sector contains space-time fermions. The mass spectrum is given by:

$$\boxed{\begin{aligned} \alpha' M^2 &= 4\left(N_R^{(\alpha)} + N_R^{(b)} - 1/2\right) \\ &= 4\left(N_L^{(\alpha)} + N_L^{(d)}\right) \end{aligned}} \qquad (40)$$

The two expressions for M^2 are consistent due to $L_0 - \bar{L}_0 = 0$.

Massless states:

$$|0\rangle_R^a \otimes b^\mu_{-1/2} |0\rangle_{NS} \; : \quad M^2 = 0 \quad \left[N_{L,R}^{(\alpha,d)} = 0, \; N_L^{(b)} = 0, \; N_R^{(b)} = 1/2 \right]$$

These contain spin 3/2 spinors in space-time. They are to ~~be~~ identified as the superpartners of $(h_{\mu\nu}, b_{\mu\nu})$.

The formulae for the NS-R sector is obtained by the interchange $L \leftrightarrow R$ in (40).

Summary :

		$M<0$	$M=0$	$M>0$
NS-NS	:	tachyon ($\|0\rangle_{NS}$)	$h_{\mu\nu}, b_{\mu\nu}, \phi$ ($b^\mu_{-1/2} \tilde{b}^\nu_{-1/2} \|0\rangle_{NS}$)	massive ($a^\mu_{-1} \tilde{a}^\nu_{-1} \|0\rangle_{NS}$)
R-R	:	x	bispinors ($\|0\rangle_R^a \times \|0\rangle_R^b$)	(massive)
R-NS	:	x	fermions $\|0\rangle_R \otimes b^\mu_{-1/2} \|0\rangle_{NS}$	(massive)

The spectrum obtained in this way has several problems:

1) It has a tachyon.
2) Although it contains space-time bosons and fermions, the spectrum is not supersymmetric.
3) The theory contains mutually non-local operators.
4) Space-time bosons may correspond to both worldsheet bosons and fermions, which is contrary to common sense. The same is true of space-time fermions.

All these problems get cured if we restrict ourselves to a subspace of states in the spectrum. This is acheived by the GSO projection which ~~___~~ removes some states from the theory. Strictly speaking, GSO naturally arises as a consequence of 1-loop modular invariance. However, for the sake of simplicity, we will try to justify it by observing that it is consistent with the restoration of space-time supersymmetry.

The GSO Projection:

The GSO projection acts independently on the right and left movers. So let us consider only the right movers explicitly.

NS-Sector:

Any state in the NS sector has either an even, or an odd number of b^{μ}_{r} excitation. The number of these excitations is measured by the fermion number operator F:

$$F^{(b)} = \sum_{r>0} b_{-r} \cdot b_r$$

$$(-1)^{F^{(b)}} = +1 \quad \text{for even fermion no.}$$

$$= -1 \quad \text{for odd fermion no.}$$

$$(-1)^{F^{(b)}} |even\rangle = |even\rangle$$
$$(-1)^{F^{(b)}} |odd\rangle = -|odd\rangle$$

The GSO projection in the NS sector corresponds to truncating the spectrum by throwing away all states with even fermion no. Thus all physical states should satisfy:

$$(NS): \quad \boxed{(-1)^{F^{(b)}} |phys\rangle = -|phys\rangle}$$

The same applies to the left moving Fock space. In the NS-NS sector, this eliminates the tachyon which is $|0\rangle_{NS}$ and $a^{\mu}_{-1} \tilde{a}^{\nu}_{-1} |0\rangle_{NS}$, but it retains $b^{\mu}_{-1/2} b^{\nu}_{-1/2} |0\rangle_{NS}$ which are the massless states of this sector.

R- Sector:

Here again, every state in the Fock space has either even, or odd number of fermionic oscillator, though there is some arbitrariness in the GSO projection.

$F^{(d)} = \sum_{n\geq 1}^{\infty} d_{-n} \cdot d_n$ counts the number of d_{-n}^{μ} oscillators for $n>0$. Therefore $(-1)^{F^{(d)}}$ measure whether the state has even or odd number of d_{-n}^{μ} oscillators for $n>0$. The number of d_0^{μ} oscillators can be counted (mod 2) as follows:

$$\underbrace{|0\rangle_R^a}_{32 \text{ components}} = \underbrace{|0\rangle_R^+ \oplus |0\rangle_R^-}_{16 \text{ components}} \quad \text{where } \Gamma_{11}|0\rangle_R^{\pm} = \pm|0\rangle_R^{\pm}$$

The 16 components in $|0\rangle_R^+$ contain even no. of d_0^{μ} oscillators, while those in $|0\rangle_R^-$ contain odd number of d_0^{μ} oscillators. Therefore

$$\Gamma_{11}|\phi\rangle = +|\phi\rangle \Rightarrow \text{even no. of } d_0^{\mu} \text{ oscillators}$$

$$\Gamma_{11}|\phi\rangle = -|\phi\rangle \Rightarrow \text{odd no. of } d_0^{\mu} \text{ oscillators.}$$

Thus the total number of d_n^{μ} oscillators, mod 2, is measured by

$$\Gamma = \Gamma_{11}(-1)^{F^{(d)}} \qquad (\text{for } n\geq 0)$$

The GSO projection corresponds to choosing states with Γ eigenvalue either $+1$, or -1, corresponding to even or odd number of d_{-n}^{μ} $(n\geq 0)$ excitations, respectively.

In the **R-sector**, there is no natural way of choosing between ~~...~~ the even or odd fermion numbers and both lead to consistent theories. Therefore the consistent GSO projections are

$$(R): \quad \boxed{\Gamma_{11} (-1)^{F^{(d)}} |p m_j s\rangle = \pm |p m_j s\rangle}$$

Let us look at the states with $F^{(d)} = 0$, i.e., $|0\rangle_R$. The GSO tells us that $|0\rangle_R$ should be either $|0\rangle_R^+$ or $|0\rangle_R^-$:

$$\Gamma_{11} |0\rangle_R = \pm |0\rangle_R$$

Therefore, the massless fermions are Majorana-Weyl fermions of definite chirality.

Type II-A and Type II-B Theories

To construct the full fock space, we have to apply the GSO projection on the left and right movers, and then put them together:

$$\left(|NS\rangle^{left} \oplus |R\rangle^{left} \right) \otimes \left(|NS\rangle^{right} \oplus |R\rangle^{right} \right)$$

NS-NS: GSO is unique for this sector and the massless states are $h_{\mu\nu}$, $b_{\mu\nu}$ and ϕ (as in the bosonic string)

Ramond sector: there are two choices here:

i) Both left and right moving R states have the same GSO projection: In particular this means that

$$\left[\Gamma_{11}\, |0\rangle_R^{left}\right] = \left[\Gamma_{11}\, |0\rangle_R^{right}\right] \qquad (\,[\;] = \text{eigenvalue})$$

Therefore, the two space-time fermions (massless)

$\left(|0\rangle_R^{left} \otimes b_{-1/2}^{\mu}|0\rangle_{NS}\right)$ and $\left(\tilde{b}_{-1/2}^{\mu}|0\rangle_{NS} \otimes |0\rangle_R^{right}\right)$ have

the same chirality. The resulting theory is called type II-B string theory which has two supersymmetries of the same chirality.

ii) The left and right moving R-states have ~~the~~ opposite GSO projections. This in particular means that

$$\left[\Gamma_{11}\, |0\rangle_R^{left}\right] = -\left[\Gamma_{11}\, |0\rangle_R^{right}\right]$$

Thus the space-time fermions have opposite chiralities. the resulting theory has two supersymmetries of opposite chirality and is called type II-A theory.

R-R : The R-R states are bi-spinors in both II-A and II-B theories:

$$\underbrace{|0\rangle_R^{a}}_{left} \otimes \underbrace{|0\rangle_R^{b}}_{right}$$

clearly, in type II-B theory, the bi-spinor is constructed out of two spinors of the same chirality in 9+1 dimensions.

In type II-A, the two spinors in the bi-spinor have opposite chiralities (note that it is only the relative chirality that matters).

Below, we will show that a bi-spinor can be decomposed into a ~~no~~ number of bosonic anti-symmetric tensor field. These fields will have even or ~~odd~~ odd numbers of indices depending on the relative chiralities of the two spinors.

The Effect of GSO on the Spin Field :

The spin field was introduced as a 32 component majorana spinor S^a, such that :

$$|0\rangle^a_R = S^a(0)\,|0\rangle_{NS}.$$

Similarly, we have $\bar{S}^a$ for the left moving sector. In terms of S^a, the GSO projection ~~mean~~ $\Gamma_{11}|0\rangle^a_R = \pm|0\rangle^a_R$ means

$$\boxed{\Gamma_{11} S = \pm S \quad , \quad \Gamma_{11}\bar{S} = \pm\bar{S}} \qquad (41)$$

In terms of S and $\bar{S}$, type II-A and type II-B are characterized as

$\underline{\underline{\text{II-A}}}$: $\Gamma_{11} S = S, \; \Gamma_{11} \tilde{S} = -\tilde{S}$

$\underline{\underline{\text{II-B}}}$: $\Gamma_{11} S = S, \; \Gamma_{11} \tilde{S} = \tilde{S}$

After GSO projection, S and $\bar{S}$ are 16 component Majorana-Weyl spinors with 8 on shell degrees of freedom in each.

<u>Summary of bosonic Massless spectrum</u>:

<u>NS-NS</u> : $b^{\mu}_{-1/2} \otimes \tilde{b}^{\nu}_{-1/2} |0\rangle_{NS} \rightarrow h_{\mu\nu}, b_{\mu\nu}, \varphi \; (G_{\mu\nu}, B_{\mu\nu}, \Phi)$

<u>R-R</u> : $|0\rangle^{left}_{R} \otimes |0\rangle^{right}_{R}$

$(h_{\mu\nu}, b_{\mu\nu}, \phi)$ appear in the vertex operators for the massless NS-NS states as polarization tensors.

To find the field content of the theory in the RR sector, we have to write a vertex operator for the corresponding massless states. This will give us the full massless bosonic field content of II-A and II-B theories. Below, we will first describe this, and then turn to T-duality between II-A and II-B theories.

R-R Fields:

Now, we describe the massless RR field content of II-A and II-B theories. These fields play a very important role in ~~the~~ recent developments in string theory.

Vertex operator:

We want to write a vertex operator for the massless RR states

$$|0\rangle_R^{left} \otimes |0\rangle_R^{right} = S(0) \otimes \tilde{S}(0) |0\rangle_{NS}$$

where, S and $\bar{S}$ are Majorana Weyl fermions (due to the GSO projection). The relevant part of the vertex operator for this state is given by

$$\boxed{V_{RR} \sim F_{ab}(p)\, S^a (i\Gamma^0 \tilde{S})^b e^{ip\cdot X}} \qquad (42)$$

Note that "$\sim$" signifies the fact that $\tilde{S}$ is a left-mover on the worldsheet and does not have any space-time connotation. Under $SO(9,1)$ Lorentz transformations, S and $\tilde{S}$ transform as Majorana-Weyl spinors. To keep V_{RR} invariant, $F_{ab}(p)$ has to transform as a bi-spinor in the appropriate way. Thus our polarization tensor is a bi-spinor. The properties of $F_{ab}(p)$ depend on the space-time chiralities of S and $\tilde{S}$ that also ~~spec~~ specify the IIA or IIB nature of the theory.

Gamma Matrix conventions

$$\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu} \quad , \quad \eta^{\mu\nu} = \{-1, +1, \cdots, +1\}$$

$$\Gamma_{11} = \Gamma_0 \Gamma_1 \cdots \Gamma_9 \qquad (\Gamma_\mu = \eta_{\mu\nu}\Gamma^\nu)$$

Γ^μ : pure imaginary $\Rightarrow (\Gamma^\mu)^* = -(\Gamma^\mu)$

$$(\Gamma^i)^T = \Gamma^i \ (i = 1, \cdots 9), \qquad (\Gamma^0)^T = -\Gamma^0.$$

$$\Gamma^0 \Gamma_\mu^\dagger \Gamma^0 = \Gamma_\mu \quad \Rightarrow \quad \Gamma^0 \Gamma_\mu \Gamma^0 = \Gamma_\mu^\dagger = -\Gamma_\mu^T$$

$$(\Gamma_{11})^2 = 1, \quad \{\Gamma_{11}, \Gamma^\mu\} = 0 \Rightarrow \quad \Gamma_{11}\Gamma^0 = -\Gamma^0\Gamma_{11}$$

$$\Gamma^{\mu_1 \cdots \mu_k} = \frac{1}{k!} \Gamma^{[\mu_1} \cdots \Gamma^{\mu_k]} = \frac{1}{k!}\left(\Gamma^{\mu_1} \cdots \Gamma^{\mu_k} \pm \text{perm.}\right)$$

The Effect of GSO on F_{ab}

The antisymmetrized products of Γ^μ form a complete basis $\{\Gamma^{\mu_1 \cdots \mu_k}\}$ in which any bi-spinor can be expanded:

$$\boxed{F_{ab} = \sum_{k=0}^{10} \frac{i^k}{k!} F_{\mu_1 \cdots \mu_k} (\Gamma^{\mu_1 \cdots \mu_k})_{ab}} \qquad (43)$$

$F_{\mu_1 \cdots \mu_k}$: rank-k anti-symmetric tensor.

$\Gamma^{\mu_1 \cdots \mu_k}$ for $k=0$ stands for 1

The GSO projection puts restrictions on F_{ab}, and hence, on $F_{\mu_1 \cdots \mu_k}$:

Type II-B Theory :

In this case, GSO implies

$$\Gamma_{11} S = S \ , \qquad \Gamma_{11} \tilde{S} = \tilde{S}$$

(Note that the overall sign of Γ_{11} is arbitrary, and we have chosen it to be $+1$)

To see the effect of this constraint on F_{ab}, insert $(1-\Gamma_{11})$ in the vertex operator:

$$F_{ab} \left[(1-\Gamma_{11}) S \right]^a (i\Gamma^0 \tilde{S})^b = 0$$

$$\Rightarrow (F_{ab}) S^a (i\Gamma^0 \tilde{S})^b = (F_{a'b} \Gamma_{11}{}^{a'}{}_{a}) S^a (i\Gamma^0 \tilde{S})^b$$

$$\Rightarrow \boxed{\Gamma_{11} F = F}$$

Now, insert $(1-\Gamma_{11})\tilde{S} = 0$ in V_{RR} :

$$F_{ab} \, S^a (i \, \Gamma^0 (1-\Gamma_{11}) \tilde{S})^b = 0$$

$$\Rightarrow \qquad F_{ab} S^a (i\Gamma^0 \tilde{S})^b = F_{ab} S^a (i\Gamma^0 \Gamma_{11} \underbrace{\Gamma^0 \Gamma^0}_{=1} \tilde{S})^b$$

$$= F_{ab} S^a i \underbrace{(\Gamma^0 \Gamma_{11} \Gamma^0)^b{}_{b'}}_{-\Gamma_{11}} (\Gamma^0 \tilde{S})^{b'}$$

$$= -F_{ab} S^a (\Gamma_{11})^b{}_{b'} (\Gamma^0 \tilde{S})^{b'}$$

$$F_{ab} S^a (i\Gamma^0 \tilde{S})^b = -(F\Gamma_{11})_{ab} (i\Gamma^0 \tilde{S})^b .$$

$$\Rightarrow \qquad \boxed{F = -F\Gamma_{11}}$$

$\therefore$ In II-B theory, all degrees of freedom are contained in a bi-spinor F_{ab} that satisfies the condition

$$\text{(II-B)} \qquad \boxed{\Gamma_{11} F = -F\Gamma_{11} = F} \qquad (44)$$

Type II-A Theory:

In this case, GSO implies

$$\Gamma_{11} S = S, \qquad \Gamma_{11} \tilde{S} = -\tilde{S},$$

and all degrees of freedom are contained in a bi-spinor F_{ab} satisfying

$$\text{(II-A)} \qquad \boxed{\Gamma_{11} F = +F\Gamma_{11} = F} \qquad (45)$$

The Effect of GSO on $F_{\mu_1 \cdots \mu_k}$

The conditions on f_{ab} can be translated into conditions on $F_{\mu_1 \cdots \mu_k}$ in terms of which f_{ab} is expanded.

This is done by imposing (44) and (45) on (43). We need the following identities:

$$\boxed{\begin{aligned} \Gamma_{11} \Gamma^{\mu_1 \cdots \mu_k} &= \frac{(-)^{[k/2]}}{(10-k)!} \epsilon^{\mu_1 \cdots \mu_{10}} \Gamma_{\mu_{k+1} \cdots \mu_{10}} \\ \Gamma^{\mu_1 \cdots \mu_k} \Gamma_{11} &= \frac{(-)^{[\frac{k+1}{2}]}}{(10-k)!} \epsilon^{\mu_1 \cdots \mu_{10}} \Gamma_{\mu_{k+1} \cdots \mu_{10}} \end{aligned}}$$

Here, $[x]$ is the integer part of x.

Type II-B :

$$\Gamma_{11} F = - F \Gamma_{11} \Rightarrow \Gamma_{11} \Gamma^{\mu_1 \cdots \mu_k} = - \Gamma^{\mu_1 \cdots \mu_k} \Gamma_{11}$$

From the expansions above, this is possible only when k is odd.

$\therefore$ II-B only contains $F_{\mu_1 \cdots \mu_k}$ for odd k, i.e.,

F_μ, $F_{\mu\nu\rho}$, $F_{\mu\nu\rho\sigma\lambda}$, ...

Type II-A :

$$\Gamma_{11} F = F \Gamma_{11} \Rightarrow \Gamma_{11} \Gamma^{\mu_1 \cdots \mu_k} = \Gamma^{\mu_1 \cdots \mu_k} \Gamma_{11} .$$

$\therefore$ II-A only contains $F_{\mu_1\cdots\mu_k}$ for even k, i.e.,

$F_0,\ F_{\mu\nu},\ F_{\mu\nu\rho\sigma},\ \cdots$

Furthermore, in both II-A and II-B, $\Gamma_{11} F = F \Rightarrow$

$$\sum_{k=0}^{10} \frac{i^k}{k!} F_{\mu_1\cdots\mu_k} \frac{(-)^{[k/2]}}{(10-k)!} \epsilon^{\mu_1\cdots\mu_{10}} \Gamma_{\mu_{k+1}\cdots\mu_{10}}$$

$$= \sum_{k=0}^{10} \frac{i^k}{k!} F_{\mu_1\cdots\mu_k} \Gamma^{\mu_1\cdots\mu_k}$$

<u>Sketch:</u>

$$F = \sum_{k=0}^{10} F_k \Gamma^{(k)}, \qquad \Gamma_{11}\Gamma^{(k)} = *\,\Gamma^{(10-k)}$$

$$\Rightarrow \quad \Gamma_{11} F = \sum_{\underset{k' = 10-k}{\underset{\downarrow}{k=0}}}^{10} F_k * \Gamma^{(10-k)} = \sum_{k=0}^{10} F_k \Gamma^{(k)} = F.$$

$$\Rightarrow \quad \sum_{k'=0}^{10} F_{10-k'} * \Gamma^{k'} = \sum_{k=0}^{10} F_k \Gamma^{(k)}.$$

$$\Rightarrow \quad \boxed{F_k = *\,F_{10-k}}$$

$\therefore$ In II-A the independent forms are $F_0, F_{\mu\nu}, F_{\mu\nu\rho\sigma}$

The rest are related to these by Hodge duality.
In II-B the independent ones are $F_\mu, F_{\mu\nu\rho}, F_{\mu\nu\rho\sigma\lambda}$.
and the 5-form is self dual:

$$\boxed{F_{(5)} = *\,F_{(5)}}$$

Eqns. of Motion for $F_{\mu_1 \cdots \mu_n}$

The eqns of motion for $F_{\mu_1 \cdots \mu_n}$ follow from the constraint $J^A_\alpha = 0$. They are:

$$(p_\mu \Gamma^\mu) F = F(p_\mu \Gamma^\mu) = 0.$$

For $F_{\mu_1 \cdots \mu_n}$, they imply

$$p^{[\mu} F^{\nu_1 \cdots \nu_n]} = p_\mu F^{\mu\nu_2 \cdots \nu_n} = 0.$$

In coordinate space,

$$\boxed{\underbrace{dF = 0}_{\text{Bianchi identity}}, \qquad \underbrace{d * F = 0}_{\text{Eqn. of motion}}}$$

$$dF = 0 \Rightarrow F_{(n)} = d\, C_{(n-1)}$$

$$\boxed{F_{\mu_1 \cdots \mu_n} = \frac{1}{(n-1)!} \partial_{[\mu_1} C_{\mu_2 \cdots \mu_n]}}$$

$C_{(n)}$ are the n form potentials and $F_{(n+1)}$ are the corresponding field strengths.

II - A : odd-form potentials. A_μ, $C_{\mu\nu\rho}$ (also $dF_0 = 0$)

II - B : even-form potentials: χ, $B'_{\mu\nu}$, $D^+_{\mu\nu\rho\sigma}$ (self dual).

Generic form of the low-energy Effective Action :

$$S = -\frac{1}{2k^2_{(10)}} \int d^{10}x \sqrt{-G} \Big[e^{-2\Phi} \left(R - 4 \nabla_\mu \Phi \nabla^\mu \Phi + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right)$$

$$+ \sum_k \frac{1}{2k!} F^2_{(k)} \Big]$$

$$F^2_{(k)} = F_{\mu_1 \cdots \mu_k} F^{\mu_1 \cdots \mu_k}.$$

For II-A : $k = 0, 2, 4$.

For II-B : $k = 1, 3$. (There is no covariant action for the self dual 5-form $F_{(5)} = * F_{(5)}$)

T- Duality in Superstring Theory

T - duality map :

consider

$\{X^\mu\} = \{X^i, X^9\}$, with $X^9(\sigma+2\pi) = X^9(\sigma) + 2\pi R n$.

T-duality

$$\left\{ X^9 = X^9_L + X^9_R, \quad \psi^9_+, \quad \psi^9_- \right\}$$

$$\downarrow$$

$$\left\{ \tilde{X}^9 = \tilde{X}^9_L + \tilde{X}^9_R, \quad \tilde{\psi}^9_+, \quad \tilde{\psi}^9_- \right\}$$

such that:

$$\boxed{\begin{array}{ll} \tilde{X}^9_R = X^9_R & \tilde{\psi}^9_- = \psi^9_- \\ \tilde{X}^9_L = -X^9_L & \tilde{\psi}^9_+ = -\psi^9_+ \end{array}}$$

The effect of T-duality on the X^9 coordinate is the same as the bosonic case:

$$m \longleftrightarrow n, \quad R \longleftrightarrow \frac{\alpha'}{R} .$$

The transformation of ψ^9_+ is required by world-sheet supersymmetry. From here, one can see that $\bar{\Gamma}_{11} \longrightarrow -\Gamma_{11}$ and GSO for the left movers ~~are~~ is reversed.

Another way of seeing this is to regard T-duality as a "parity" in space time. Then

$$S \longrightarrow S \qquad \bar{S} \longrightarrow \Gamma^9 \Gamma^{11} \bar{S} .$$

Note that the effect of parity ($X^9 \to -X^9$) on a space-time fermion (Ψ) is

$$\Psi \longrightarrow \Gamma^9 \Gamma^{11} \Psi$$

[This can be seen from the covariance of $\Gamma^\mu \partial_\mu \Psi = 0$]

In our case, the parity is not performed on X^9, but rather on its left moving component on the worldsheet: $X_L \to -X_L$. Therefore, only the space-time fermion coming from the left moving sector of the worldsheet theory ($\bar{S}$) should undergo a parity-like transformation under T-duality:

$$S \longrightarrow S \quad , \quad \bar{S} \longrightarrow \tilde{\bar{S}} = \Gamma^9 \Gamma_{11} \bar{S} .$$

One can easily check that if $\Gamma_{11} \bar{S} = \pm 1$, then $\Gamma_{11} \tilde{\bar{S}} = \mp 1$.

let $P^9 = \Gamma^9 \Gamma_{11}$. $\tilde{\bar{S}} = P^9 \bar{S}$, $\tilde{\Gamma}_{11} = P^9 \Gamma_{11} (P^9)^{-1}$.

Then, $\Gamma_{11} \bar{S} = \bar{S} \Rightarrow \tilde{\Gamma}_{11} \tilde{\bar{S}} = \tilde{\bar{S}}$, but, $\Gamma_{11} \tilde{\bar{S}} = -\tilde{\bar{S}}$

We can write a vertex operator in terms of $\tilde{\bar{S}}$ (instead of $\bar{S}$) corresponding to a state for which winding and momentum has been interchanged. The bispinor polarizations are then related by

$$\boxed{\tilde{F} = F \Gamma^9 \Gamma_{11}}$$

In terms of the k-form components, this translates into:

$$\tilde{F}_{\mu_1 \cdots \mu_k} = - F_{9\mu_1 \cdots \mu_k}$$

$$\tilde{F}_{9\mu_1 \cdots \mu_k} = F_{\mu_1 \cdots \mu_k}$$

where $\mu_i \neq 9$. ~~Note~~ Hence T-duality interchanges II-A and II-B theories by interchanging even and odd form fields. Note that the change in the dimensions of k-form fields is simply related to dimensional reduction along X^9. This is consistent with the fact that in 1+8 dimensions, II-A + II-B have ~~the~~ the same field content.

What does T-duality between two theories mean?

Consider a theory T_R on a circle of radius R (along X^9) with vertex operators V_R constructed out of $(X^\mu, \psi^\mu, S, \bar{S})$, also containing $e^{ik\cdot X}$ such that $k^9 = 0$. In this theory, vertex operators $(\tilde{V}_R)$ obtained from V_R by the replacement $(X^\mu, \psi^\mu, S, \bar{S})_R \longrightarrow (\tilde{X}^\mu, \tilde{\psi}^\mu, S, \tilde{\bar{S}})_R$ are also physical vertex operators corresponding to interchanging momentum and winding modes without changing the radius R. Correlation functions involving V_R and those involving $\tilde{V}_R$ are different. Let us take T_R to be type II B theory, so that $\Gamma_{11}\bar{S} = \bar{S}$ $(\Gamma_{11}\tilde{\bar{S}} = -\tilde{\bar{S}})$.

Now, consider a theory $T_{1/R}$ on a circle of radius $1/R$ with vertex operators $V_{1/R}$'s constructed out of $(X^\mu, \psi^\mu, S, \bar{S})_{1/R}$ where $\Gamma_{11}\bar{S} = -\bar{S}$ (type II-A). Then, a correlation function involving $V_{1/R}$'s in the theory $T_{1/R}$ is the same as a correlation in T_R involving $\tilde{V}_R$'s. The correspondence between $\tilde{V}_R$ and $V_{1/R}$ is the obvious one: $V_{1/R}$ is obtained from $\tilde{V}_R$ by regarding $(\tilde{X}^\mu, \tilde{\psi}^\mu, S, \tilde{\bar{S}})_R$ of the theory T_R as the $(X^\mu, \psi^\mu, S, \bar{S})_{1/R}$ of the theory $T_{1/R}$. Note that the chirality of $\bar{S}$ is matched consistently. $(\Gamma_{11})_R \bar{S}_R = \bar{S}_R$, $(\Gamma_{11})_R \tilde{\bar{S}}_R = -\tilde{\bar{S}}_R$ and $(\Gamma_{11})_{1/R}\bar{S}_{1/R} = -\bar{S}_{1/R}$ so one can consistently make the correspondence $\tilde{\bar{S}}_R \leftrightarrow \bar{S}_{1/R}$ (here "‾" denotes the left moving worldsheet sector).

S-duality in Type II-B Theory

Field content of the type II-B theory:

NS-NS: $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ.

R-R: $D^{+}_{\mu\nu\rho\sigma}$ $\tilde{B}_{\mu\nu}$ χ

$$\begin{array}{ccc} D^{+}_{\mu\nu\rho\sigma} & \tilde{B}_{\mu\nu} & \chi \\ \uparrow & \uparrow & \uparrow \\ C_{(4)} & C_{(2)} & C_{(0)} \end{array}$$

Field strengths:

$$\left.\begin{array}{ll} F_{(5)} = dC_{(4)}, & F_{(5)} = * F_{(5)} \\ F_{(3)} = dC_{(2)}, & F_{(1)} = dC_{0} \end{array}\right\} dF_{(k)} = 0$$

Eqns. of motion for the R-R fields:

$$d * F_{(k)} = 0 \Rightarrow \boxed{\partial_{\mu_1} F^{\mu_1 \cdots \mu_k} = 0}$$

Clearly, these are the generalizations of Maxwell's equations, $\partial_\mu F^{\mu\nu} = 0$.

Except for the $F_{(5)}$, we can write an effective action that reproduces the equations of motion. However, the self-duality condition on $F_{(5)}$ ~~can be~~ cannot be obtained from ~~an effective action~~. covariant action. Thus, we consider the special case of $F_{(5)} = 0$, and write an effective action for the remaining fields:

$$S_{eff.} = \frac{1}{2k^2_{(10)}} \int d^{10}x \sqrt{-G} \Big[e^{-2\phi}\left(R - 4\nabla_\mu\phi \nabla^\mu \phi + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}\right) + \frac{1}{2}\nabla_\mu \chi \nabla^\mu \chi + \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} \Big]$$

where, $$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \cdots$$

$$F_{\mu\nu\rho} = \partial_\mu \tilde{B}_{\nu\rho} + \cdots$$

Excep for some factors, $(B_{\mu\nu}, \phi)$ and $(\tilde{B}_{\mu\nu}, \chi)$ appear almost on the same footing in S_{eff}. The theory has an $SL(2,Z)$ symmetry ~~unless~~ that rotates these two doubles into each other. **Note** that this is not a perturbative symmetry of the theory. We will first describe these transformations and then explain their consequences.

The $SL(2,Z)$ symmetry is manifest after we rewrite S_{eff} in a more appropriate form:

First, $$g_{\mu\nu} = e^{-\phi/2} G_{\mu\nu} \qquad (g_{\mu\nu} : \text{Einstein metric}).$$

Introduce: $$\lambda = \chi + i e^{-\phi}.$$

$$B = \begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix}, \text{ where } B^{(1)}_{\mu\nu} = B_{\mu\nu}, \quad B^{(2)}_{\mu\nu} = \tilde{B}_{\mu\nu}.$$

$$H = dB.$$

$$\mathcal{M} = e^{\phi} \begin{pmatrix} |\lambda|^2 & \chi \\ \chi & 1 \end{pmatrix}$$

Then,

$$S_{eff} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \Big[R + \frac{1}{4} tr(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}) - \frac{1}{12} H^T \mathcal{M} H \Big]$$

This is invariant under

$$\mathcal{M} \to \Lambda \mathcal{M} \Lambda^T, \quad B \to (\Lambda^T)^{-1} B$$

To keep the form of $\mathcal{M}$ invariant, $\Lambda \in SL(2,\mathbb{R})$.

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow$$

$$\boxed{\begin{gathered} \lambda \to \frac{a\lambda + b}{c\lambda + d} \\ B^{(1)} \to d B^{(1)} - c B^{(2)}, \qquad B^{(2)} \to a B^{(2)} - b B^{(1)} \end{gathered}} \quad (46)$$

$$(ad - bc = 1)$$

1) This transformation can relate weak and strong couplings. This is easiest to see when $\chi = 0$. Then $SL(2,\mathbb{R})$ contains an element under which

$$e^{-\phi} \to e^{+\phi} \qquad (g \sim e^{\phi})$$

2) Even if we start with $B^{(2)} = 0$, after this transformation $B^{(2)} \neq 0$.

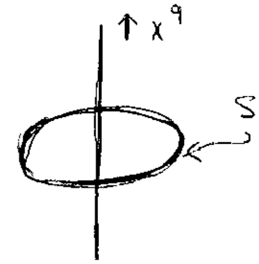
3) In quantum theory, $SL(2,\mathbb{R})$ is broken to $SL(2,\mathbb{Z})$ due to quantization of $B_{\mu\nu}$ charge. (charge quantization applies to 1-brane and 5-brane charges).

Clearly, $SL(2,\mathbb{Z})$ is a non-perturbative symmetry and its correctness cannot be checked in perturbation theory. Therefore, to check if (46) is really a symmetry of the full type II-B theory, one has to check its consequences. We briefly describe this below:

Fundamental Strings and RR charged strings

$B^{(1)}_{\mu\nu}$ can have non-trivial configurations for which

$$\int_{S^7} * H^{(1)} = q.$$



This is recognized as the charge per unit length for a 1-brane. The 1-dimensional object to which $B^{(1)}_{\mu\nu}$ couples is the fundamental string as can be seen from the way $B^{(1)}_{\mu\nu}$ appears in the world sheet action:

$$\int d^2\sigma \; \epsilon^{\alpha\beta} B_{\mu\nu} \, \partial_\alpha X^\mu \partial_\beta X^\nu = \int B_{\mu\nu} \, dX^\mu \wedge dX^\nu$$

where $\sigma^\alpha = (\tau, \sigma)$. Compare this with the way an electromagnetic field couples to a charged particle in the action.

$$\int \vec{A} \cdot \vec{v} \, d\tau \rightarrow \int A_\mu \frac{dx^\mu}{d\tau} d\tau \equiv \int d\tau \, A_\mu \partial_\tau X^\mu = \int A_\mu dX^\mu .$$

In fact in II-B one can find a classical soln. to the eqns of motion obtained from S_{eff} with $\chi = B^{(2)}_{\mu\nu} = 0$, which can be identified as the field configuration around a fundamental string.

Soln:

$$ds^2 = G_{\mu\nu}\, dx^\mu dx^\nu$$

$$= A^{-3/4}\left[-dt^2 + (dx^1)^2\right] + A^{1/4}\, d\vec{x}\cdot d\vec{x}$$

$$(\vec{x} : x^2, \dots x^9).$$

$$B^{(1)}_{01} = e^{2\phi} = A^{-1} \qquad (\text{other } B^{(1)}_{\mu\nu} = 0),$$

where $\quad A(r) = 1 + \dfrac{Q}{3r^6} \qquad (r^2 = \vec{x}\cdot\vec{x})$

Then, $Q = B^{(1)}_{\mu\nu}$ charge carried by the string:

$$\int_{S^7} * H = Q \qquad (B^{(2)} = \chi = 0)$$

This soln is singular at $r \to 0$ This can be interpreted as the presence of the ~~[illegible]~~ fundamental string at $r=0$. Therefore, the ~~[illegible]~~ existence of this singular soln is in accordance with the existence of fundamental strings.

Let us now look at the effect of a special $SL(2,Z)$ transformation on this solution:

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad d=0 \quad b=-1, \quad c=1 \quad a=0.$$

then,

$$e^{\phi} \to e^{-\phi}, \qquad B^{(1)} \longrightarrow B^{(2)}.$$

After this transf. we have another classical soln with $B^{(1)}=0$ but $B^{(2)} \neq 0$. This soln. carries charge

Q for the RR field $B^{(2)}_{\mu\nu}$. Howere, string theory does not contain any perturbative feature that could account for the $B^{(2)}_{\mu\nu}$ charge. It now appears as if the theory contains a dual string which is the source of the RR charge in exactly the same way that the fundamental string was the source of the NS-NS charge for $B^{(1)}_{\mu\nu}$. When on string is strongly coupled, its dual is weakly coupled.

More general $SL(2,Z)$ transformations produce configuration in which both $B^{(1)}$ and $B^{(2)}$ have non-zero charges. One can also estimate the tension (mass per unit length) of these solutions. It appears that these solutions describe bound states of "m" fundamental strings with "n" RR-charged string. If $SL(2,Z)$ is a true symmetry of the theory, then such bound states should exist. This actually turns out to be the case: R-R charged solitons are described by D-branes and bound states of fundamental strings with D-strings do exist and agree with the predictions of S-duality.

5-branes

Another class of objects in II-B theory are 5-branes. They are the magnetic duals of strings.

NS, 5-brane ← fundamental string

RR- 5-brane (D-5 brane) ← D-string.

Type II-A and M-theory

Field content of type II-A theory:

NS-NS: $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ $\quad [(\mu,\nu): 0,\cdots,9]$
R-R : A_μ, $C_{\mu\nu\rho}$.

Field content of supergravity in D = 11:

g_{MN} $\quad c_{MNP}$ $\quad [(M,N): 0,\cdots,10]$.

Field content of II-A from 11-D supergravity
$M = \{\mu, 10\}$.

$$\phi = g_{1010}, \quad G_{\mu\nu} = g_{\mu\nu}, \quad B_{\mu\nu} = c_{\mu\nu 10}$$

$$A_\mu = g_{\mu 10}, \quad C_{\mu\nu\rho} = c_{\mu\nu\rho}$$

More precisely, at R: radius of 11^{th} dim. Then,

$$R = \lambda^{2/3}, \quad \text{where, } \lambda = e^{\phi}$$

$\therefore$ $R \to \infty \Rightarrow$ strong coupling in II-A.

More evidence: D0 branes in IIA $\leftrightarrow$ KK modes in 11-D.

Consider M theory compactification to D = 9 on T^2.
$M^{(11)}$ on T^2 = IIA on $S^1_{(R)} \longrightarrow$ IIB on $S^1_{(1/R)}$.
The $SL(2,\mathbb{Z})$ of II-B is the $SL(2,Z)$ of the compactification (T^2).