Conformal Dimension Formula and the OPE

We have derived a formula for computing the conformal dimensions of primary fields ϕ,

$$i[Q_{\varepsilon^-}, \phi] = (h\, \partial_- \varepsilon^- + \varepsilon^- \partial_-)\, \phi$$

and a similar formula for the left moving sector (for definitness, we consider closed strings). We now relate this formula to the structure of the $T\phi$ operator product expansion. To acheive this, we have to work on the Euclidean worldsheet and represent σ-integrals as contour integrals.

Q_ε as a contour integral: consider the Euclidean continuation of the worldsheet theory,

$$\tau = -i\tau_E \qquad , \quad (\tau_E : \text{real})$$

Then, $\quad \sigma^- = \tau - \sigma = -i(\tau_E - i\sigma) = -i\,\sigma_E^-$

and

$$z = e^{2\pi i(\tau - \sigma)/l} = e^{2\pi(\tau_E - i\sigma)/l} = e^{2\pi\sigma_E^-/l}$$

At fixed τ, $\quad d\sigma = i\, d\sigma_E^- \quad$ so that

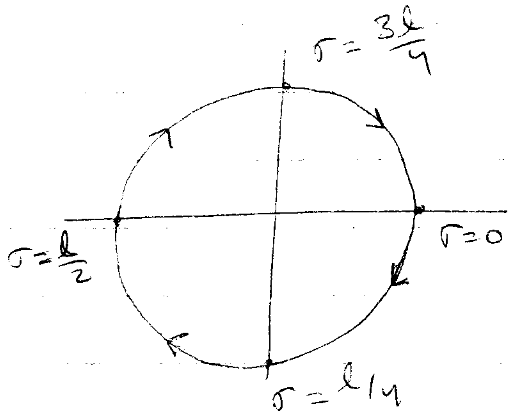
$$d\sigma = \frac{-l}{2\pi i}\,\frac{dz}{z}$$

$$\int_0^\ell d\sigma = \frac{-\ell}{2\pi i}\int_{C_{cw}}\frac{dz}{z} = \frac{\ell}{2\pi i}\int_{C_{ccw}}\frac{dz}{z}$$

C_{cw} : clockwise contour

C_{ccw} : counter-clockwise contour

Note that σ varying from 0 to 2π maps to a clockwise contour in the z-plane (since $z = (*)\, e^{-2\pi i\sigma/\ell}$)



$z = (*)\, e^{-2\pi i\sigma/\ell}$

Also,

$$T_{--} = \partial_- X \cdot \partial_- X = i^2 \partial_{\sigma_E^-} X \cdot \partial_{\sigma_E^-} X = -\left(\frac{\partial z}{\partial \sigma_E^-}\right)^2 \partial_z X \cdot \partial_z X$$

$$= -\left(\frac{2\pi}{\ell}\right)^2 z^2 T_{zz}$$

and $\qquad \varepsilon^- \equiv \delta\sigma^- = -i\,\delta\sigma_E^- = -i\,\frac{\ell}{2\pi}\,\frac{\delta z}{z}$

Hence,

$$Q_{\varepsilon^-} = T\int_0^\ell d\sigma\, \varepsilon(\sigma^-)\, T(\sigma^-) = -T\int_{C_{ccw}} \frac{\cancel{\ell}}{\cancel{2\pi} i}\frac{dz}{\cancel{z}}\; \frac{\cancel{\ell}}{\cancel{2\pi} i}\frac{\delta z}{\cancel{z}}\left(\frac{2\pi}{\cancel{\ell}}\right)^{\cancel{2}} \cancel{z^2}\, T_{zz}$$

$$= 2\pi i\, T\int_{C_{ccw}}\frac{dz}{2\pi i}\,\delta z\, T_{zz}$$

Using the notation $C = C_{ccw}$, $\delta z = \zeta(z)$, we finally have,

$$\boxed{Q_{\varepsilon^-} = \frac{i}{\alpha} \int_C \frac{dz}{2\pi i}\, \zeta(z)\, T_{zz}(z)}$$

In terms of the Fourier modes on the cylinder, we have

$$T_{zz}(z) = -\left(\frac{\ell}{2\pi}\right)^2 \frac{1}{z^2}\, T(\sigma^-)$$

$$= -\left(\frac{\ell}{2\pi}\right)^2 \left(\frac{2\pi}{\ell^2 T}\right) \sum_m L_m\, z^{-m-2}$$

and,

$$\zeta(z) = i\left(\frac{2\pi}{\ell}\right) z\, \varepsilon^-(\sigma^-) = i\left(\frac{2\pi}{\ell}\right) \sum_n z^{-n+1}\, \varepsilon^-_n$$

Therefore,

$$Q_{\varepsilon^-} = \frac{2\pi}{\ell} \int_C \frac{dz}{2\pi i} \sum_{m,n} \varepsilon^-_n L_m\, z^{-m-n-1}$$

Note that in order to get the correct expression for Q_{ε^-}, i.e.,

$$Q_{\varepsilon^-} = \frac{2\pi}{\ell} \sum_m \varepsilon^-_{-m} L_m$$

one has to interpret $\int_C dz/2\pi i$ as a contour integral in the z-plane, around $z = 0$. The integral then picks up the simple pole corresponding to $m+n+1 = 1$, leading to the desired result.

Note that Q_{ε^-} in terms of $T_{zz}(z)$ and $\zeta(z)$ has the same form as Q_{ε^-} in terms of $T(\sigma^-)$ and ε^-. In this sense it is invariant under $\sigma^- \rightarrow z$,

$$Q_{\varepsilon^-} = T\int_0^\ell d\sigma\, \varepsilon(\sigma^-)\, T_{--}(\sigma^-) = T\int_C dz\, \zeta(z)\, T_{zz}(z)$$

From this, we expect that the conformal dimension formula on the z-plane has the same form as that on the cylinder:

Conformal Dimension Formula on the z-plane

Let $\phi(\sigma^-)$ denote a conformal primary of weight h on the cylinder. Then

$$\phi(\sigma^-) = \left(\frac{\partial\tilde{\sigma}^-}{\partial\sigma^-}\right)^h \tilde{\phi}(\tilde{\sigma}^-)$$

where $\tilde{\sigma}^- = \tilde{\sigma}^-(\sigma^-)$ (From now on we will supress the left-moving sector). Let us denote the corresponding field on the z-plane by $\phi(z)$. Since $\sigma^- \to z$, or $\sigma_E^- \to z = \exp(2\pi\, \sigma_E^-/\ell)$ is a conformal transformation, we have,

$$\phi(\sigma^-) = \left(\frac{\partial z}{\partial\sigma^-}\right)^h \phi(z)$$

Now, the conformal transformation $\sigma^- \to \tilde{\sigma}^-$ on the cylinder induces the corresponding transformation $z \to \tilde{z}$ on the z-plane. It then follows that

$$\phi(z) = \left(\frac{\partial\tilde{z}}{\partial z}\right)^h \tilde{\phi}(\tilde{z})$$

$$\phi(z) = \left(\frac{\partial\sigma^-}{\partial z}\right)^h \phi(\sigma^-) = \left(\frac{\partial\sigma^-}{\partial z}\right)^h \left(\frac{\partial\tilde{\sigma}^-}{\partial\sigma^-}\right)^h \tilde{\phi}(\tilde{\sigma}^-) = \left(\frac{\partial\tilde{\sigma}^-}{\partial z}\right)^h \tilde{\phi}(\tilde{\sigma}^-)$$

$$= \left(\frac{\partial\tilde{\sigma}^-}{\partial\tilde{z}}\right)^h \left(\frac{\partial\tilde{z}}{\partial z}\right)^h \tilde{\phi}(\tilde{\sigma}^-) = \left(\frac{\partial\tilde{z}}{\partial z}\right)^h \tilde{\phi}(\tilde{z})$$

The infinitesimal form of this transformation can be easily worked out: corresponding to $\sigma^- \to \sigma^- + \varepsilon(\sigma^-)$, we have

$$z \to z + \zeta(z) \qquad \text{with } \zeta(z) = i\frac{2\pi}{\ell} z \varepsilon^- = \frac{2\pi}{\ell} z\, \delta\sigma_\varepsilon^-$$

Then, for $\tilde{z} = z + \zeta(z)$, we get

$$\delta\phi(z) = \tilde{\phi}(z) - \phi(z) = -\left(h\,\partial_z\zeta + \zeta\partial_z\right)\phi(z)$$

To express this in terms of the generator Q_ε, note that

$$\frac{\partial\sigma^-}{\partial z} = \frac{\ell}{2\pi i}\frac{1}{z}, \qquad \zeta(z) = \frac{2\pi i}{\ell} z\varepsilon^-$$

then,

$$\delta\phi(z) = -\left(h\,\partial_z\left(\frac{2\pi i}{\ell} z\varepsilon^-\right)\phi(z) + \left(\frac{2\pi i}{\ell} z\varepsilon^-\right)\partial_z\left[\left(\frac{\ell}{2\pi i}\frac{1}{z}\right)^h \phi(\sigma^-)\right]\right)$$

$$= -\left(\cancel{\frac{2\pi i}{\ell} h\varepsilon^-\left(\frac{\partial\sigma^-}{\partial z}\right)^h}\phi(\sigma^-) + h\frac{2\pi i}{\ell} z \frac{\ell}{2\pi i}\frac{1}{z}\partial_-\varepsilon^-\left(\frac{\partial\sigma^-}{\partial z}\right)^h\phi(\sigma^-)\right.$$

$$\left. + \frac{2\pi i}{\ell} z\varepsilon^-\left(\frac{\ell}{2\pi i}\frac{1}{z}\right)^h \frac{\ell}{2\pi i}\frac{1}{z}\partial_-\phi(\sigma^-) - \cancel{\frac{2\pi i}{\ell} z\varepsilon^-\left(\frac{\ell}{2\pi i}\right)^h \frac{h}{z^{h+1}}}\phi(\sigma^-)\right)$$

$$= -\left(\frac{\partial\sigma^-}{\partial z}\right)^h \left(h\,\partial_-\varepsilon^- + \varepsilon^-\partial_-\right)\phi(\sigma^-)$$

$$= -\left(\frac{\partial\sigma^-}{\partial z}\right)^h i\left[Q_\varepsilon, \phi(\sigma^-)\right]$$

$$= -i\left[Q_\varepsilon, \phi(z)\right]$$

Thus the conformal dimension formula on the z-plane is

$$\boxed{i\left[Q_{\varepsilon}, \phi(z)\right] = \left(h\, \partial_z \zeta(z) + \zeta(z)\partial_z\right) \phi(z)}$$

which has the same form as the corresponding equation on the cylinder, and with the same Q_ε appearing in both (the basic reason for this is that the gauge fixed worldsheet action has the same form on the plane and on the cylinder.

Conformal Dimension Formula in terms of the OPE

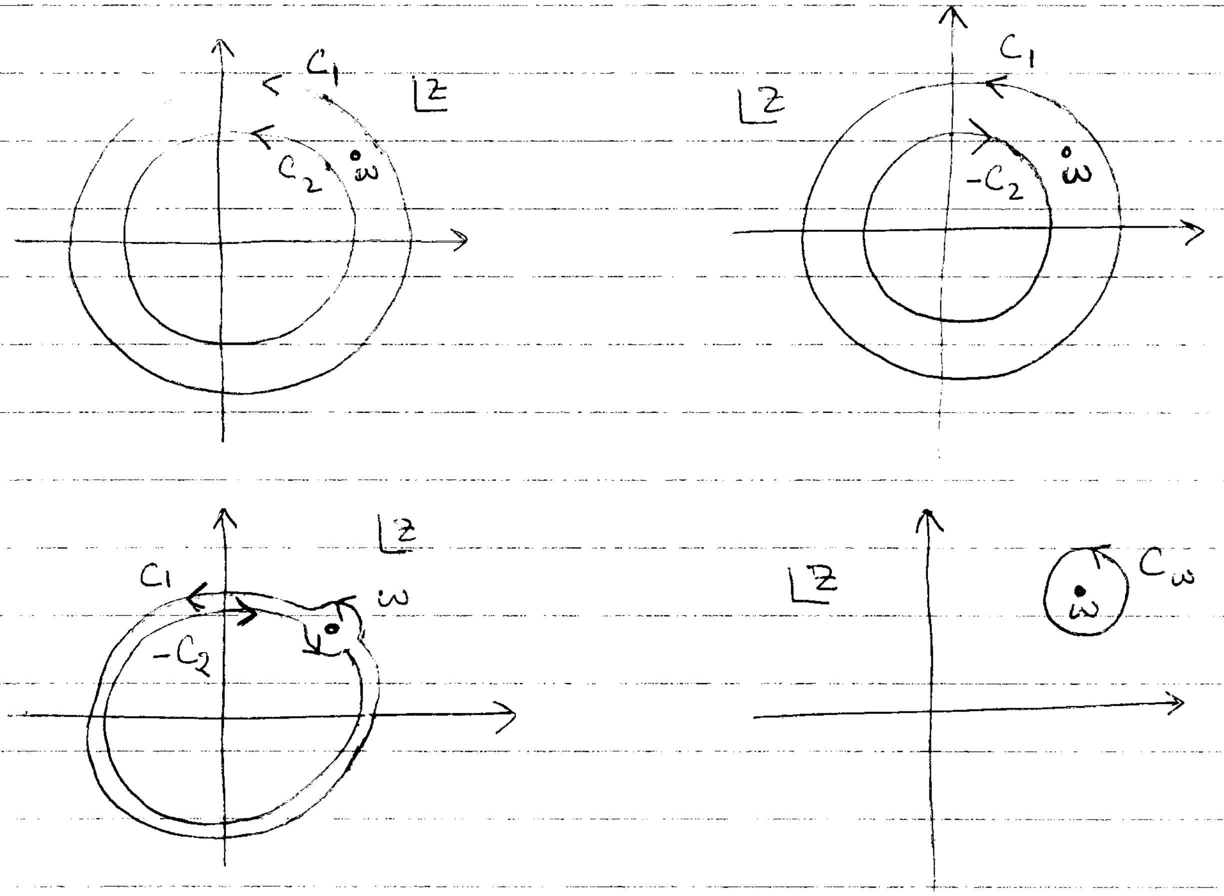
We are now finally in a position to recast the conformal dimension formula for $\phi(z)$ in terms of the $T(z)\,\phi(w)$ OPE which greatly simplifies the calculation of $(h, \bar{h})$ for various fields.

Remember that Q_ε can be represented as a contour integral around the origin, $z = 0$. Then,

$$\left[Q_{\epsilon}, \phi(w)\right] = T\int_{C_1} dz\, \zeta(z)\, T(z)\, \phi(w) - T\int_{C_2} dz\, \zeta(z)\, \phi(w)\, T(z)$$

we are able to choose different contours C_1 and C_2 since Q_ε does not depend on the shape of the contour. C_1 and C_2 can be chosen such on C_1, $|z| > |w|$ while on C_2, $|z| < |w|$. Then, using the "Radial ordering" notation, $R(T(z)\,\phi(w)) = T(z)\,\phi(w)\,\theta(|z|-|w|) + \phi(w)\,T(z)\,\theta(|w|-|z|)$, we can write,

$$[Q_\epsilon, \phi(w)] = T\int_{C_1} dz\, \zeta(z)\, R(T(z)\,\phi(w)) - T\int_{C_2} dz\, \zeta(z)\, R(T(z)\,\phi(w))$$



Replacing the counter-clockwise contour C_2 by the clockwise contour $-C_2$, the second term changes sign,

$$[Q_\epsilon, \phi(w)] = T\left(\int_{C_1} + \int_{-C_2}\right) dz\, \zeta(z)\, R(T(z)\,\phi(w))$$

Away from $z=w$ the contours can be deformed so that they cancel each other as the integrand only has poles at $z=w$ (this is evident from the operator product expansion of $R(T(z)\,\phi(w))$). The only contribution then comes from the contour $C_w = C_1 - C_2$ around $z=w$,

$$\boxed{\left[Q_\varepsilon\,,\ \phi(w)\right] = \frac{i}{\alpha'}\int_{C_w}\frac{dz}{2\pi i}\,\zeta(z)\ R\big(T(z)\ \phi(w)\big)}$$

This is the contour integral representation of the commutator. It is an important tool in conformal field theory allowing us to use the techniques of complex integration to do quantum field theory calculations. If $\phi(w)$ is a conformal primary of weight h, then

$$\int_{C_w}\frac{dz}{2\pi i}\ \zeta(z)\ R\big(T(z)\ \phi(w)\big) = -\alpha'\left(h\,\partial_w\zeta(w) + \zeta(w)\,\partial_w\right)\phi(w)$$

$$= -\alpha'\int_{C_w}\frac{dz}{2\pi i}\left[\frac{h\,\zeta(z)\,\phi(w)}{(z-w)^2} + \frac{\zeta(z)\,\partial_w\phi(w)}{(z-w)}\right]$$

where we have used the standard formula

$$\int_{C_w}\frac{dz}{2\pi i}\ \frac{f(z)}{(z-w)^{n+1}} = \frac{1}{n!}\,\partial_w^n\,f(w)$$

to express the right-hand-side too as a contour integral. From this one can easily read off the singular terms (in $z-w$) of the $R(T(z)\ \phi(w))$ OPE as,

$$\boxed{R\big(T(z)\ \phi(w)\big) = -\alpha'\left[\frac{h}{(z-w)^2}\,\phi(w) + \frac{1}{(z-w)}\,\partial_w\,\phi(w) + \cdots\right]}$$

For any conformal primary $\phi(w)$, the $T(z)\,\phi(w)$ OPE has this structure, in particular the number h multiplies $\frac{\phi(w)}{(z-w)^2}$. Now, we have already seen that given a field $\phi(w)$ as a function of the X^μ's, the $T(z)\,\phi(w)$ OPE can be evaluated in a straightforward way using the Wick expansion of the radially ordered expression. Comparing the OPE evaluated in this way with the RHS of the above eqn. we can see if $\phi(z)$ is a conformal primary and also read off its conformal dimension h. When ϕ is a complicated enough function of X^μ, this method of evaluating h is much more convenient than calculating the QFT commutators.

There is also a similar formula for the left-hand sector (or the antiholomorphic sector) of the theory,

$$R\left(\bar{T}(\bar{z})\,\phi(\bar{w})\right) = -\alpha'\left[\frac{\bar{h}}{(\bar{z}-\bar{w})^2}\,\phi(\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\,\partial_{\bar{w}}\phi(\bar{w}) + \cdots\right].$$

As an application, let us consider some of the OPE's that we have already computed, e.g.,

$$R\left(T(z)\,\partial_w X^\rho(w)\right) = -\alpha'\left[\frac{1}{(z-w)^2}\,\partial_w X^\rho(w) + \frac{1}{(z-w)}\,\partial_w^2 X^\rho(w) + \cdots\right]$$

$$R\left(T(z) :e^{ik\cdot X(w)}:\right) = -\alpha'\left[\frac{\alpha' k^2/4}{(z-w)^2}\,e^{ik\cdot X(w)} + \frac{1}{(z-w)}\,\partial_w e^{ik\cdot X(w)}\right]$$

Clearly, for $\partial_w X^\rho$, $h=1$, $\bar{h}=0$ and for $:e^{ik\cdot X}:$, $h = \frac{\alpha' k^2}{4} = \bar{h}$.

As another application of these techniques one can easily derive the Virasoro algebra including the central charge term: First note that the prensence of the term $\frac{D/2}{(z-w)^4}$ in the $R(T(z)\,T(w))$ OPE shows that $T(w)$ is not a primary field. Now, the relation between commutators and contour integrals gives

$$[Q_\zeta, T(w)] = \frac{i}{\alpha'}\oint \frac{dz}{2\pi i}\, \zeta(z)\, R\big(T(z)\,T(w)\big)$$

$$= -i\oint \frac{dz}{2\pi i}\, \zeta(z)\left(\frac{-\alpha' D/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w)\right)$$

$$= -i\left(\frac{-\alpha' D/2}{3!}\partial_w^3 \zeta(w) + 2\,\partial_w \zeta(w)\, T(w) + \zeta(w)\,\partial_w T(w)\right)$$

where we have used the $T(z)\,T(w)$ OPE and then performed the contour integral. Now, remember that,

$$Q_\epsilon = \frac{2\pi}{\ell}\sum_m \epsilon_{-m} L_m, \qquad T(z) = \frac{-1}{2\pi T}\sum_m L_m z^{-m-2}$$

$$\zeta(z) = i\left(\frac{2\pi}{\ell}\right)\sum_n z^{-2+1}\,\epsilon_n$$

substituting in the above expression and comparing the coefficients of z^{-n-2} on both sides gives,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}\,n(n^2-1)\,\delta_{n+m,0}$$