Some OPE Calculations

Note 1: In mapping from the Euclidean world-sheet (τ, σ) to the complex plane $(z, \bar{z})$, we have seen that the Euclidean time τ parametrizes the radius on the complex plane ($|z| = e^{\tau}$). Hence "time-ordering" on the cylinder is equivalent to "radial-ordering" on the complex plane (i.e., $\tau_2 > \tau_1 \Rightarrow e^{\tau_2} > e^{\tau_1}$). We denote radial-ordering by $R(\cdots)$ and the ~~two~~ concepts are used synonymously.

Note 2: Very often we are interested in the sterss-energy tensor $T_{\alpha\beta}$ in terms of the $(z, \bar{z})$ coordinates:

$$T_{--}(\sigma^-) = :\partial_- X \cdot \partial_- X: = \left(\frac{\partial z}{\partial \sigma^-}\right)^2 :\partial_z X \cdot \partial_z X: = \left(i \frac{2\pi}{\ell} z\right)^2 :\partial_z X \, \partial_z X:$$

$$= -\left(\frac{2\pi}{\ell}\right)^2 z^2 \, T_{zz}(z)$$

$$\boxed{T_{zz}(z) = :\partial_z X \cdot \partial_z X:}$$

Let us now compute the operator product expansions of some radially ordered products of fields on the complex plane:

① $$R\left(T_{zz}(z)\, \partial_w X^\rho(w)\right) = R\left(:\eta_{\mu\nu}\, \partial_z X^\mu \partial_z X^\nu: \partial_w X^\rho(w)\right)$$

$$= \eta_{\mu\nu} \langle \partial_z X^\mu(z) \partial_w X^\rho(w)\rangle \partial_z X^\nu(z) + \eta_{\mu\nu} \langle \partial_z X^\nu(z)\, \partial_w X^\rho(w)\rangle \partial_z X^\mu(z)$$

$$= -\frac{\alpha'}{2}\frac{1}{(z-w)^2}\left(\eta_{\mu\nu}\eta^{\mu\rho}\partial_z X^\nu(z) + \eta_{\mu\nu}\eta^{\nu\rho}\partial_z X^\mu(z)\right)$$

$$= \frac{-\alpha'}{(z-w)^2}\,\partial_z X^\rho(z)$$

Expanding $\partial_z X^\rho(z)$ around $z=w$ (which is the argument of the 2nd factor in the radially ordered product),

$$\partial_z X^\rho(z) = \partial_w X^\rho(w) + (z-w)\partial_w^2 X^\rho(w) + \frac{(z-w)^2}{2!}\partial_w^3 X^\rho(w) + \cdots$$

and retaining only the non-regular terms one gets,

$$\boxed{R\left(T_{zz}(z)\,\partial_w X^\rho(w)\right) = -\alpha'\frac{1}{(z-w)^2}\partial_w X^\rho(w) - \alpha'\frac{1}{(z-w)}\partial_w^2 X^\rho(w) + \cdots}$$

(2) $R\left(T_{zz}(z)\,T_{ww}(w)\right) = R\left(\eta_{\mu\nu} :\partial_z X^\mu(z)\partial_z X^\nu(z): \;\eta_{\rho\sigma} :\partial_w X^\rho(w)\partial_w X^\sigma(w):\right)$

$$= 2\eta_{\mu\nu}\eta_{\rho\sigma}\langle\partial_z X^\mu(z)\partial_w X^\rho(w)\rangle\langle\partial_z X^\nu(z)\partial_w X^\sigma(w)\rangle$$

$$+ 4\eta_{\mu\nu}\eta_{\rho\sigma}\langle\partial_z X^\mu(z)\partial_w X^\rho(w)\rangle :\partial_z X^\nu(z)\partial_w X^\sigma(w):$$

On substituting for $\langle\partial_z X^\mu \partial_w X^\rho\rangle$ and expanding $\partial_z X(z)$ around $z=w$ one gets the TT OPE as

$$\boxed{R\left(T_{zz}(z)\, T_{\omega\omega}(\omega)\right) = \alpha'^2 \frac{D/2}{(z-\omega)^4} - \alpha' \frac{2}{(z-\omega)^2} T_{\omega\omega}(\omega) - \frac{\alpha'}{(z-\omega)} \partial_\omega T_{\omega\omega}(\omega) + \cdots}$$

(3) $R\left(T_{zz}(z) : e^{ik.X(\omega)} :\right) = R\left(\eta_{\mu\nu} : \partial_z X^\mu(z)\, \partial_z X^\nu(z) :: \sum_n \frac{(i)^n}{n!}\left(k_\rho X^\rho(\omega)\right)^n :\right)$

$$= \sum_n \left\{ 2n \frac{(i)^n}{n!} k_\rho \eta_{\mu\nu} \langle \partial_z X^\mu(z)\, X^\rho(\omega)\rangle : \partial_z X^\nu(z) \left(k\cdot X(\omega)\right)^{n-1} : \right.$$

$$+ 2n(n-1) \frac{(i)^n}{n!} \eta_{\mu\nu} k_\rho k_\sigma \langle \partial_z X^\mu(z)\, X^\rho(\omega)\rangle \langle \partial_z X^\nu(z)\, X^\sigma(\omega)\rangle$$

$$\left. \times :\left(k\cdot X(\omega)\right)^{n-2} : \right\}$$

On using $\langle \partial_z X^\mu(z)\, X^\nu(\omega)\rangle = \frac{-\alpha' \eta^{\mu\nu}}{2} \frac{1}{z-\omega}$ and expanding $\partial_z X^\mu(z)$ around $z=\omega$ we get,

$$\boxed{R\left(T_{zz}(z) : e^{ik.X(\omega)} :\right) = -\alpha' \left(\frac{\alpha' k^2/2}{(z-\omega)^2} e^{ik.X(\omega)} + \frac{1}{z-\omega} \partial_\omega e^{ik.X(\omega)} \right)}$$

In this example we have suppressed the $\bar{\omega}$ dependence of $X^\mu(\omega, \bar{\omega})$ as it does not affect the calculation.