Propagators and Operator Product Expansions

We will now describe certain Conformal Field Theory technics that make it very easy to deal with vertex operators and perform quantum computations in string theory. As in ordinary quantum field theory, all this is based on the notion of propagators or 2-point functions. Below, we compute these first for the closed and then for the open string theory.

The $(z, \bar{z})$ Coordinates

For simplicity, let us use the following notation:

$$\boxed{z = e^{2\pi i \sigma^-/\ell}, \qquad \bar{z} = e^{2\pi i \sigma^+/\ell}} \tag{1}$$

Then, for closed strings, we have

$$X^\mu = X_L^\mu(\bar{z}) + X_R^\mu(z)$$

$$X_R^\mu(z) = \frac{1}{2}x^\mu - \frac{i}{4\pi T} p^\mu \ln z + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} z^{-n} \tag{2}$$

$$X_L^\mu(\bar{z}) = \frac{1}{2}x^\mu - \frac{i}{4\pi T} p^\mu \ln \bar{z} + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} \bar{z}^{-n}$$

A comment on the nature of z and $\bar{z}$: As introduced above, these are not complex conjugates of each other. However, we can make an Euclidean continuation by writing

$$\tau = -i\bar{\tau}$$

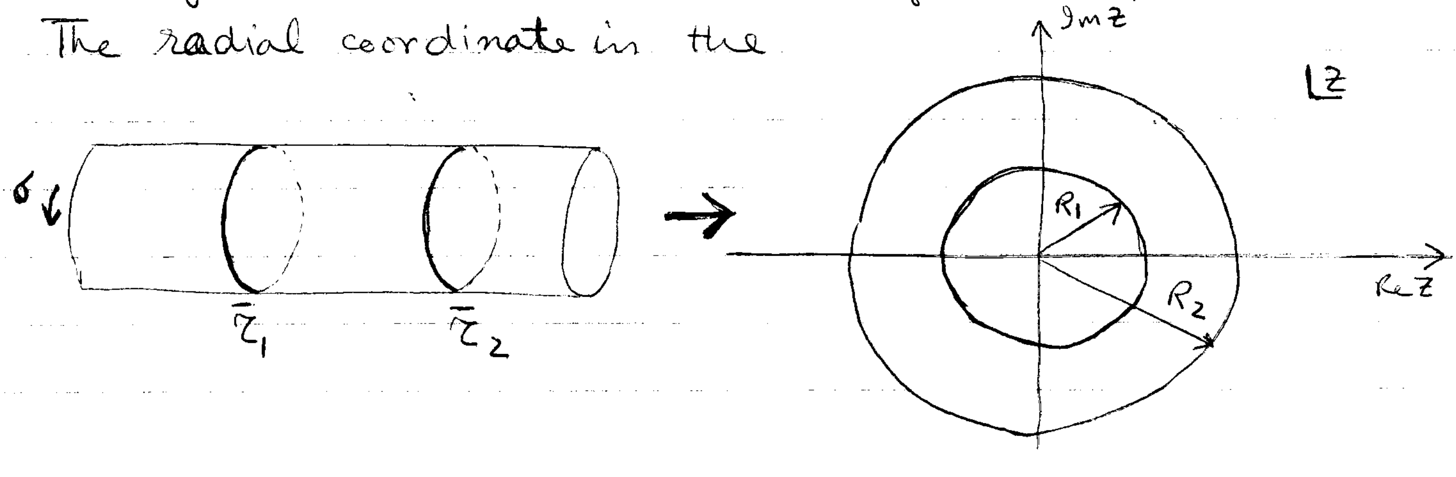
where, $\bar{\tau}$ is now regarded as a real variable. In terms of $\bar{\tau}$,

$$\boxed{z = e^{2\pi(\bar{\tau} - i\sigma)/\ell}, \quad \bar{z} = e^{2\pi(\bar{\tau}+i\sigma)/\ell}} \quad (3)$$

and are related by complex conjugation. In terms of $\bar{\tau}$ the worldsheet has an Euclidean metric,

$$ds^2 = -d\tau^2 + d\sigma^2 \rightarrow ds^2 = d\bar{\tau}^2 + d\sigma^2.$$

It is in the Euclidean formulation of the theory that the full power of conformal field theory can be used to perform string theory computations. The worldsheet is now the complex plane parametrized by z and $\bar{z}$ and we can use the technics of complex analysis. $\bar{\tau} = -\infty$ is the origin of the complex plane. The radial coordinate in the



z-plane is a measure of Euclidean world sheet time $\bar{\tau}$:

$$R = |z| = e^{2\pi\bar{\tau}/\ell}$$

A closed string at time $\bar{\tau}_1$ is represented by a circle of radius $R_1 = e^{2\pi\bar{\tau}_1/\ell}$ in the z-plane with $\theta = 2\pi\sigma/\ell$ as the angular coordinate. Time evolution from $\bar{\tau}_1$ to $\bar{\tau}_2$ on cylindrical worldsheet corresponds to the circle R_1 growing to one with radius R_2. Below, we will often refer to this Euclidean picture to make proper sense of our results (Although we will not discuss it here, it can be argued that such analytic continuations do not affect the physics of the problem). Note that $z = e^{2\pi(\bar{\tau}-i\sigma)/\ell}$ is a conformal transformation from $\bar{\tau}+i(-\sigma)$ to $z = z_1 + iz_2$ which is an invariance of the theory (at the classical level as well in quantum theory for $D = 26$)

After this digression we turn to the computation of the propagator in closed string theory:

Closed String Propagator

It is by now clear that we are dealing with a quantum field theory in 2 dimensions. In QFT, when dealing with products of field operators, a powerful tool is the "Wick expansion" which involves the notion of the "propagators" or "two-point functions" of the fields. The propagator for $X^\mu(\sigma,\tau)$ that appears in the corresponding Wick expansion is given by

$$\boxed{\langle X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')\rangle = T\big(X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau')\big) - :X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau'):}$$

where T is the time ordering symbol,

$$T\big(X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau')\big) = X^\mu(\sigma,\tau) X^\nu(\sigma',\tau')\, \theta(\tau-\tau') + X^\nu(\sigma',\tau') X^\mu(\sigma,\tau)\, \theta(\tau'-\tau)$$

$$\big(\theta(\tau) = 1 \text{ for } \tau>0, \quad \theta(\tau)=0 \text{ for } \tau<0 \big)$$

and $:\ :$ stands for normal ordering. Writing X^μ as $X^\mu = X^\mu_0 + X^\mu_- + X^\mu_+$ where X^μ_0 contains the zero-mode part x^μ_0, X^μ_- the raising operators $\alpha^\mu_{-m}, \tilde{\alpha}^\mu_{-m}$ $(m>0)$ and X^μ_+ the lowering operators α^μ_m, $\tilde{\alpha}^\mu_m$, p^μ, it is easy to verify that $T(X^\mu X^\nu) - :X^\mu X^\nu:$ contains only commutators of fields and hence is a c-number (not an operator as one might naively expect).
One may therefore write a more convenient expression for the propagator as

$$\boxed{\langle X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau')\rangle = \langle 0|\, T\big(X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau')\big)|0\rangle - \langle 0|{:}X^\mu(\sigma,\tau)\, X^\nu(\sigma',\tau'){:}|0\rangle}$$

The only contribution to $\langle 0|{:}(\ \ ){:}|0\rangle$ comes from the zero-mode part of the fields (use ${:}p^\mu x^\nu{:} = x^\nu p^\mu$ and $p^\mu|0\rangle = 0$),

$$\langle 0|{:}X^\mu(\sigma,\tau)\; X^\nu(\sigma',\tau'){:}|0\rangle = \langle 0|x^\mu x^\nu|0\rangle$$

To evaluate $\langle 0|T(\ \ )|0\rangle$, note that,

$$\begin{aligned}\langle 0|\, X^\mu X^\nu|0\rangle &= \langle 0|(X^\mu_L + X^\mu_R)(X^\nu_L + X^\nu_R)|0\rangle \\ &= \langle 0|\, X^\mu_L X^\nu_L + X^\mu_R X^\nu_R + X^\mu_L X^\nu_R + X^\mu_R X^\nu_L\,|0\rangle.\end{aligned}$$

where $X^\mu_R(z)$ and $X^\mu_L(\bar z)$ are given by (2). Explicity,

$$\begin{aligned}\langle 0|X^\mu_R(z)\; X^\nu_R(z')|0\rangle &= \frac{1}{4}\langle 0|\, x^\mu x^\nu|0\rangle - \frac{i}{8\pi T}\ln z\,\langle 0|\,p^\mu x^\nu|0\rangle \\ &\quad - \frac{1}{4\pi T}\sum_{\substack{m\neq 0\\ n\neq 0}}\frac{1}{mn}\langle 0|\alpha^\mu_n\; \alpha^\nu_m|0\rangle\, z^{-n} z'^{-m}\end{aligned}$$

$$= \frac{1}{4}\langle 0|x^\mu x^\nu|0\rangle - \frac{\eta^{\mu\nu}}{8\pi T}\ln z + \frac{1}{4\pi T}\sum_{\substack{n>0\\ m>0}}\langle 0|\alpha^\mu_n\; \alpha^\nu_{-m}|0\rangle\frac{z^{-n} z'^{m}}{nm}$$

where we have used $p^\nu|0\rangle = 0$, $[x^\mu, p^\nu] = i\eta^{\mu\nu}$, $\alpha^\mu_m|0\rangle$ for $m>0$ and $\langle 0|\alpha^\mu_n = 0$ for $n<0$. Also, using $[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}$, we have

$$\langle 0|\, X^\mu_R(z)\; X^\nu_R(z')|0\rangle = \frac{1}{4}\langle 0|x^\mu x^\nu|0\rangle - \frac{\eta^{\mu\nu}}{8\pi T}\ln z + \frac{1}{4\pi T}\sum_{n>0}\cancel{n}\,\eta^{\mu\nu}\,\frac{z^{-n} z'^{n}}{\cancel{n}\, n}$$

The series can be summed over using

$$\ln(1-y) = -\sum_{n=1}^{\infty} \frac{y^n}{n}$$

so that

$$\langle 0| X_R^\mu(z)\, X_R^\nu(z')|0\rangle = \frac{1}{4}\langle 0|x^\mu x^\nu|0\rangle - \frac{\eta^{\mu\nu}}{4\pi T}\left(\frac{\ln z}{2} - \sum_{n=1}^{\infty}\frac{(z'/z)^n}{n}\right)$$

$$= \frac{1}{4}\langle 0|x^\mu x^\nu|0\rangle - \frac{\eta^{\mu\nu}}{4\pi T}\left(\frac{\ln z}{2} + \ln\left(1 - z'/z\right)\right)$$

or

$$\langle 0| X_R^\mu(z)\, X_R^\nu(z')|0\rangle = \frac{1}{4}\langle 0|x^\mu x^\nu|0\rangle + \frac{\alpha'}{4}\eta^{\mu\nu}\ln z - \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-z')$$

$$\left(\alpha' = \frac{1}{2\pi T}\right)$$

Similarly,

$$\langle 0| X_L^\mu(\bar z)\, X_L^\nu(\bar z')|0\rangle = \frac{1}{4}\langle 0|x^\mu x^\nu|0\rangle + \frac{\alpha'}{4}\eta^{\mu\nu}\ln \bar z - \frac{\alpha'}{2}\eta^{\mu\nu}\ln(\bar z - \bar z')$$

$$\langle 0| X_R^\mu(z)\, X_L^\nu(\bar z')|0\rangle = -\frac{\alpha'}{4}\eta^{\mu\nu}\ln z$$

$$\langle 0| X_L^\mu(\bar z)\, X_R^\nu(z')|0\rangle = -\frac{\alpha'}{4}\eta^{\mu\nu}\ln \bar z$$

Putting these together gives

$$\langle 0| X^\mu(z,\bar z)\, X^\nu(z',\bar z')|0\rangle = \langle 0|x^\mu x^\nu|0\rangle - \frac{\alpha'\eta^{\mu\nu}}{2}\left(\ln(z-z') + \ln(\bar z - \bar z')\right)$$

$$= \langle 0|x^\mu x^\nu|0\rangle - \frac{\alpha'\eta^{\mu\nu}}{2}\ln\left((z-z')(\bar z - \bar z')\right)$$

~~S~~imilarly,

$$\langle 0|X^\nu(z',\bar{z}')\, X^\mu(z,\bar{z})|0\rangle = \langle 0|x^\mu x^\nu|0\rangle - \frac{\alpha' \eta^{\mu\nu}}{2} \ln\left((z'-z)(\bar{z}'-\bar{z})\right)$$

$$= \langle 0|x^\mu x^\nu|0\rangle - \frac{\alpha' \eta^{\mu\nu}}{2} \ln\left((z-z')(\bar{z}-\bar{z}')\right)$$

Note that $\langle 0|X^\nu(z',\bar{z}')\, X^\mu(z,\bar{z})|0\rangle = \langle 0|X^\mu(z,\bar{z})\, X^\nu(z',\bar{z}')|0\rangle$ and hence,

$$\boxed{\langle 0|T\left(X^\mu(z,\bar{z})\, X^\nu(z',\bar{z}')\right)|0\rangle = \langle 0|x^\mu x^\nu|0\rangle - \frac{\alpha' \eta^{\mu\nu}}{2} \ln\left((z-z')(\bar{z}-\bar{z}')\right)}$$

Finally, substituting these back into the expression for the 2-point function gives

$$\boxed{\langle X^\mu(z,\bar{z})\ X^\nu(z',\bar{z}')\rangle = -\frac{\alpha' \eta^{\mu\nu}}{2}\left(\ln(z-z') + \ln(\bar{z}-\bar{z}')\right)} \quad (6)$$

From this one easily computes,

$$\langle \partial_z X^\mu(z,\bar{z})\ X^\nu(z',\bar{z}')\rangle = \frac{-\alpha' \eta^{\mu\nu}}{2}\, \frac{1}{z-z'}$$

$$\boxed{\langle \partial_z X^\mu(z,\bar{z})\, \partial_{z'} X^\nu(z',\bar{z}')\rangle = \frac{1}{2}\alpha' \eta^{\mu\nu} \frac{1}{(z-z')^2}} \quad (7)$$

Similarly,

$$\boxed{\langle \bar{\partial}_z X^\mu(z,\bar{z})\, \bar{\partial}_{z'} X^\nu(z',\bar{z}')\rangle = \frac{1}{2}\alpha' \eta^{\mu\nu} \frac{1}{(\bar{z}-\bar{z}')^2}} \quad (8)$$

Open String Propagator

For open strings we have the expansion

$$\boxed{X^\mu(z,\bar{z}) = x^\mu + \frac{2}{\ell T}p^\mu \tau + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}\left(z^{-n}+\bar{z}^{-n}\right)} \qquad (9)$$

where, $\quad \tau = \dfrac{-i\ell}{4\pi}\left(\ln z + \ln \bar{z}\right)$

The calculation proceeds in parallel with the closed string case. Using the normal ordering conventions $:p^\mu x^\nu: = x^\nu p^\mu$ and the standard definition for the α_n^μ, we have

$$\langle 0| :X^\mu(z,\bar{z})\, X^\nu(z',\bar{z}'): |0\rangle = \langle 0|\, x^\mu x^\nu |0\rangle$$

and

$$\langle 0|\, X^\mu(z,\bar{z})\, X^\nu(z',\bar{z}')|0\rangle = \langle 0| x^\mu x^\nu|0\rangle - \frac{2}{\ell T} i\eta^{\mu\nu}\tau$$

$$+ \frac{1}{4\pi T}\sum_{\substack{m=1\\n=1}}^{\infty}\left(\frac{z^{-n}+\bar{z}^{-n}}{n}\right)\left(\frac{z'^m+\bar{z}'^m}{m}\right)\langle 0|\alpha_n^\mu \alpha_{-m}^\nu|0\rangle$$

$$= \langle 0|x^\mu x^\nu|0\rangle - \frac{\eta^{\mu\nu}}{2\pi T}\left(\ln z + \ln \bar{z}\right) + \frac{1}{4\pi T}\sum_{n=1}^{\infty}\frac{\eta^{\mu\nu}}{n}\left(z^{-n}+\bar{z}^{-n}\right)\left(z'^n+\bar{z}'^n\right)$$

$$= \langle 0|x^\mu x^\nu|0\rangle - \frac{\eta^{\mu\nu}}{2\pi T}\left(\ln z + \ln\bar{z}\right) + \frac{1}{4\pi T}\eta^{\mu\nu}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{z'}{z}\right)^n + \left(\frac{z'}{\bar{z}}\right)^n + \left(\frac{\bar{z}'}{z}\right)^n + \left(\frac{\bar{z}'}{\bar{z}}\right)^m\right]$$

$$= \langle 0|x^\mu x^\nu|0\rangle - \frac{\eta^{\mu\nu}}{2\pi T}\left(\ln z + \ln\bar{z}\right) - \frac{\eta^{\mu\nu}}{4\pi T}\left(\ln\left(1-\frac{z'}{z}\right) + \ln\left(1-\frac{z'}{\bar{z}}\right) + \ln\left(1-\frac{\bar{z}'}{z}\right) + \ln\left(1+\frac{\bar{z}'}{\bar{z}}\right)\right)$$

where we have used

$$\ln(1-y) = -\sum_{n=1}^{\infty}\frac{y^n}{n}$$

Using the properties of the ln function, this finally gives

$$\langle 0| X^\mu(z,\bar{z})\, X^\nu(z',\bar{z}')|0\rangle = \langle 0|x^\mu x^\nu|0\rangle - \frac{\eta^{\mu\nu}}{4\pi T}\Big(\ln(z-z')(\bar{z}-\bar{z}') + \ln(z-\bar{z}')(\bar{z}-z')\Big)$$

From the right hand side it is clear that

$$\langle 0| X^\mu(z,\bar{z})\, X^\nu(z',\bar{z}')|0\rangle = \langle 0| X^\nu(z',\bar{z}')\, X^\mu(z,\bar{z})|0\rangle$$

Hence $\langle 0| T\big(X^\mu(z,\bar{z})\, X^\nu(z',\bar{z}')\big)|0\rangle = \langle 0| X^\mu(z,\bar{z})\, X^\nu(z',\bar{z}')|0\rangle$

The propagator defined by equation (5) now becomes

(10)

$$\boxed{\langle X^\mu(z,\bar{z})\, X^\nu(z',\bar{z}')\rangle = \frac{-\eta^{\mu\nu}}{4\pi T}\left\{\ln\left((z-z')(z-\bar{z}')\right) + \ln\left((\bar{z}-\bar{z}')(\bar{z}-z')\right)\right\}}$$

It is also customary to denote the correletion functions by a "contraction",

$$\langle X^\mu X^\nu\rangle \equiv \underbracket{X^\mu \; X^\nu}$$

Wick's Theorem & Operator Product Expansion

Wick's theorem enables us to expand the time-ordered product of a set of fields in terms of normal ordered products and contractions as

$$
\begin{aligned}
T(X_1\, X_2\, X_3 \cdots X_n) = \; & :X_1 X_2 \cdots X_n: \\
& + :\underbrace{X_1\, X_2} \cdots X_n: + \text{(other 1-contraction terms)} \\
& + :\overbrace{X_1\, X_2}\, \underbrace{X_3\, X_4} \cdots X_n: + \text{(all other 2-contraction terms)} \\
& + \cdots
\end{aligned}
$$

Of course the "contracted" fields reduce to c-numbers and are not affected by the normal ordering, which only applies to the non-contracted fields. So $:\underbrace{X_1 X_2} \cdots X_n:$ really stands for $\langle X_1 X_2 \rangle :X_3 \cdots X_n:$.

If a group of operators under T is already normal ordered, e.g., $T(:X_1 X_2 \cdots X_m: X_{m+1} \cdots X_n)$, then the expansion proceeds as above except that contractions between elements of the normal ordered set are zero and will not appear in the expansion,

$$
\begin{aligned}
T(:X_1 \cdots X_m: X_{m+1} \cdots X_n) = \; & :X_1 X_2 \cdots X_n: + :X_1 \underbrace{\cdots X_n: X_{m+1}} \cdots X_n: \\
& + \cdots \quad + :X_1 \cdots X_n \underbrace{X_{m+1}\, X_{m+2}} \cdots X_n: + \cdots
\end{aligned}
$$

In the following we suppress the $\bar{z}$ index for simplicity.

Now, consider two operators $O_1(z)$ and $O_2(w)$ which are functions of $X^\mu(z)$ and $X^\mu(w)$ respectively, and all products of X^μ's appearing in them are normal ordered: $O_1(z) = :f_1(X(z)):$, $O_2(w) = :f_2(X(w)):$ If we expand $T(O_1(z)\,O_2(w))$ using Wick's theorem, the expansion will only contain contractions involving $X^\mu(z)$ and $X^\nu(w)$ but not $\underbrace{X(z)\,X(z)}$ and $\underbrace{X(w)\,X(w)}$. Schematically, it will have the form

$$T\left(O_1(z)\,O_2(w)\right) = \sum_{ij}\sum_n \langle X(z)\,X(w)\rangle^n :g_{1i}(X(z))\;g_{2j}(X(w)):$$

where g_{1i} and g_{2i} are other functions of $X^\mu(z)$ and $X^\mu(w)$. For z very close to w, we can expand $X(z)$ around the point $z=w$ in a Taylor series. This will enable us to expand $g_{1i}(X(z))$ in terms of functions of $X(w)$ and its derivatives,

$$g_{1i}(X(z))\,g_{2j}(X(w)) = \sum_n (z-w)^n\,F_{nij}(X(w))$$

Thus the normal ordered products appearing in the expansion of $T(O_1(z)\,O_2(w))$ can all be written in terms of functions of operators at w. This shows that in general, we have the following structure

$$\boxed{T\left(O_i(z)\,O_j(w)\right) = \sum_k C_{ijk}(z-w)\;O_k(w)}$$

This is the operator product expansion. The $C_{ijk}(z-w)$ are called the structure functions. From simple dimensional analysis it follows that if, under the scalings $z \to \lambda z$ (and $w \to \lambda w$), O_i have scaling dimensions h_i $\left(O_i(\lambda z) = \lambda^{-h_i} O_i(z) \right)$, then

$$C_{ijk}(z-w) = (z-w)^{h_k - h_i - h_j} \, C_{ijk}$$

where C_{ijk} are constants. For given O_i and O_j the $C_{ijk}(z-w)$ and the set of operators $O_k(w)$ can be explicitly computed by following the procedure described above. Some examples will be worked out below.

Most often we are interested in OPE's in the limit of $z \to w$. In this case terms that involve positive powers of $(z-w)$ go to zero while ones with negative powers survive. In fact as we will see later, even when $(z-w)$ is not small, our calculations will only be sensitive to terms that are singular in the limit $(z-w) \to 0$. Because of this, we only retain terms with negative powers of $(z-w)$.