String Quantization (old Covariant Approach)

In the 2-dimensional worldsheet theory, the dynamical variables are the fields $X^\mu(\sigma,\tau)$ and the canonically conjugate momenta,

$$\pi_\nu(\sigma,\tau) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\nu)} = T\partial_\tau X_\nu(\sigma,\tau)$$

The theory is quantized by imposing Equal Time Commutation Relations (ETCR's) on the field variables,

$$\boxed{\begin{array}{ll} [X^\mu(\sigma,\tau),\ \pi^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma') & \text{— (D)} \\ [X^\mu(\sigma,\tau),\ X^\nu(\sigma',\tau)] = 0, \quad [\pi^\mu(\sigma,\tau),\ \pi^\nu(\sigma',\tau)] = 0 & \text{— (E)} \end{array}}$$

From the 2d point of view, where X^μ are simply fields, this corresponds to 2^{nd} quantization. However from the space-time point of view, where X^μ are coordinate functions and π^ν are momentum densities, this is more like 1^{st} quantization. This makes more sense since one ends up with a single string description (as opposed to a many string description).

Using $\pi^\mu = T\partial_\tau X^\mu$, the ETCR's can be written as,

$$\boxed{\begin{array}{ll} [\partial_+ X^\mu(\sigma,\tau),\ \partial'_+ X^\nu(\sigma',\tau)] = \frac{i}{2T}\eta^{\mu\nu}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') & \text{— (A)} \\ [\partial_- X^\mu(\sigma,\tau),\ \partial'_- X^\nu(\sigma',\tau)] = \frac{-i}{2T}\eta^{\mu\nu}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') & \text{— (B)} \\ [\partial_+ X^\mu(\sigma,\tau),\ \partial'_- X^\nu(\sigma',\tau)] = [\partial_- X^\mu(\sigma,\tau),\ \partial'_+ X^\nu(\sigma',\tau)] = 0 & \text{— (C)} \end{array}}$$

These equations are now used to derive the commutation relations for the α's.

Closed strings: In this case,

$$\partial_+ X^\mu = \frac{\sqrt{\pi}}{\ell\sqrt{T}} \sum_n \bar{\alpha}_n^\mu e^{-2\pi i n(\tau+\sigma)/\ell}, \quad \partial_- X^\mu = \frac{\sqrt{\pi}}{\ell\sqrt{T}} \sum_n \alpha_n^\mu e^{-2\pi i n(\tau-\sigma)/\ell}$$

Since, $0 \leq \sigma \leq \ell$, we have an expression for $\delta(\sigma-\sigma')$ in terms of the same basis functions that appear in $\partial_\pm X^\mu$,

$$\delta(\sigma-\sigma') = \frac{1}{\ell} \sum_m e^{-2\pi i m(\sigma-\sigma')/\ell}$$

Substituting these in (A) gives,

$$\frac{\pi}{\ell^2 T} \sum_{m,n} [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu]\, e^{-2\pi i (m+n)\tau/\ell}\, e^{-2\pi i (m\sigma + n\sigma')/\ell} = \frac{\pi}{\ell^2 T} \sum_m m e^{-2\pi i m(\sigma-\sigma')/\ell}$$

Since the RHS does not depend on τ, it follows that the commutator on the LHS should be non zero only for $m+n=0$, i.e, $[\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = [\bar{\alpha}_m^\mu, \bar{\alpha}_{-m}^\nu]\,\delta_{m+n,0}$. With $m+n=0$, $m\sigma + n\sigma' = m(\sigma-\sigma')$. Now, comparing the terms on the LHS and RHS, one gets $[\bar{\alpha}_m^\mu\, \bar{\alpha}_{-m}^\nu] = \eta^{\mu\nu} m$, so that $[\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}$. Relation (B) leads to a similar equation for α_n^μ. Relation (C) implies that α_n^μ and $\bar{\alpha}_m^\nu$ commute. Using the commutation relations for α_n^μ's and $\bar{\alpha}_n^\mu$'s thus obtained in (D) one also obtains $[x^\mu, p^\nu] = i\eta^{\mu\nu}$, with all commutators involving x^μ or p^μ and the α's vanishing.

To summarize, for the closed string one obtains the commutation relations,

$$\boxed{[x^\mu, p^\nu] = i\eta^{\mu\nu}, \; [\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}, \; [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}}$$

with all other commutators vanishing.

Open Strings: In this case, $\partial_\pm X^\mu = \frac{\sqrt{\pi}}{l\sqrt{T}} \sum_n \alpha^\mu_n e^{-2\pi i n(\tau \pm \sigma)/l}$

but since $0 \leqslant \sigma \leqslant l/2$, the delta function cannot be expanded in terms of $\exp(2\pi i n\sigma/l)$, unless the range of σ is doubled. That this can be done follows from the fact that for open strings, $X^\mu(-\sigma) = X^\mu(\sigma)$. Then as $\sigma \to -\sigma$ one has $\partial_- X^\mu \to \partial_+ X^\mu$, $\frac{\partial}{\partial\sigma} \to -\frac{\partial}{\partial\sigma}$ and for the new σ, $-\frac{l}{2} \leqslant \sigma \leqslant 0$. Then the commutation relation (B) becomes, under $\sigma \to -\sigma$, $\sigma' \to -\sigma'$

$$\text{(B)} \Rightarrow \left[\partial_+ X^\mu(\sigma,\tau), \partial'_+ X^\nu(\sigma',\tau)\right] = \frac{i}{2T}\eta^{\mu\nu}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma'), \; -\frac{l}{2} \leqslant \sigma, \sigma' \leqslant 0$$

$$\text{(A)}: \left[\partial_+ X^\mu(\sigma,\tau), \partial'_+ X^\nu(\sigma',\tau)\right] = \frac{i}{2T}\eta^{\mu\nu}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma'), \; 0 \leqslant \sigma, \sigma' \leqslant \frac{l}{2}$$

This shows that (A) and (B) can be combined into a single equation of the same form over $-\frac{l}{2} \leqslant \sigma, \sigma' \leqslant \frac{l}{2}$:

$$\left[\partial_+ X^\mu(\sigma,\tau), \partial'_+ X^\nu(\sigma',\tau)\right] = \frac{i}{2T}\eta^{\mu\nu}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma'), \; -\frac{l}{2} \leqslant \sigma, \sigma' \leqslant \frac{l}{2}$$

Now a simple computation similar to the closed string case leads to,

$$\boxed{[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}}$$

with all others commutators vanishing.

Open String Spectrum :

Let us first look at the case of open strings. Now that α^μ_m, and hence X^μ, have been promoted to operators, we have to find the space of states on which they act. The physical state of the system is then represented by a state in this space. The state space can be constructed using the standard procedure. We start with,

$$\boxed{[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \quad , \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}}$$

and define the number operators N_m:

$$N_m = \alpha_{-m}\cdot\alpha_m \qquad\qquad m \geqslant 1$$

These are hermitian. Consider an eigen state of N_m:

$$N_m|s_m\rangle = s_m|s_m\rangle$$

Then, using the basic commutation relation, it follows that

$$N_m(\alpha^\mu_m|s_m\rangle) = (s_m - m)(\alpha^\mu_m|s_m\rangle)$$

$$N_m(\alpha^\mu_{-m}|s_m\rangle) = (s_m + m)(\alpha^\mu_{-m}|s_m\rangle)$$

Thus, for $m \geqslant 1$, α^μ_{-m} raises and α^μ_m lowers the eigen values of N_m. The state with the lowest N_m eigen value is the "vacuum" or the "ground state" $|0\rangle$. It is annihilated by all lowering operators α^μ_m $m \geq 1$:

$$\alpha^\mu_m |0\rangle = 0 \qquad \forall\, m \geq 1.$$

Other states are created by applying α^μ_{-m}'s on $|0\rangle$ and are characterized by their N_m eigenvalues. Since p^μ commutes with all α's and N_m's, the states are also labeled by their p^μ eigenvalues, in particular, the ground state is $|0, k\rangle$,

$$p^\mu |0,k\rangle = k^\mu |0,k\rangle.$$

If α^μ_{-m} acts on $|0,k\rangle$ $i_{(\mu,m)}$ times, then a general state can be characterized in terms of the set of integers $\{i_{(\mu,m)}\}$,

$$|\{i_{(\mu,m)}\}, k\rangle = \prod_{m\geq 1}^{\infty} \prod_{\mu=0}^{D-1} \left(\alpha^\mu_{-m}\right)^{i_{(\mu,m)}} |0,k\rangle$$

Defining a total number operator N,

$$N = \sum_{m \geq 1} N_m \,,$$

one has

$$N|\{i_{(\mu,m)}\}, k\rangle = \sum_{\mu, m} \left(m\, i_{(\mu,m)}\right) |\{i_{(\mu,m)}\}, k\rangle$$

<u>**Norm**</u>: In this way, one can construct all possible states in the theory. However some of these states have negative norm which signals the breakdown of the

probabilistic interpretation of quantum theory and hence its inconsistency. For example, the norm of the state $\alpha^{\mu}_{-m}|0\rangle$ is

$$\langle 0|\alpha^{\mu}_{m}\alpha^{\mu}_{-m}|0\rangle = \langle 0|0\rangle \eta^{\mu\mu} m .$$

For $\mu=0$, $\eta^{00}=-1$ and hence the problem.

Since the negative norm states are associated with α^{0}_{-m}, their existence is clearly a manifestation of the classical problem of string oscillations in the time direction. This in turn was an artifact of the manifest Lorentz covariant formulation of the theory. At the classical level the problem was resolved by the Virasoro constraints. We now have to find a way of implementing these constraints at the quantum level.

Normal Ordering: In an expression containing a product of α^{μ}_{n}'s, their ordering is of no consequence in the classical theory. In the quantum theory where α^{μ}_{n} and α^{μ}_{-n} no longer commute, their ordering in an expression acquires significance. This results in an ambiguity while promoting classical expressions to quantum ones. The ambiguity is resolved by following an ordering prescription. The "normal ordering" prescription states that in any expression containing products of α^{μ}_{m}'s, these should be arranged such that all lowering operators stand to the right of raising operators. The normal ordering of an expression f is denoted by $:f:$

<u>The Virasoro generators</u>: Using normal ordering, in the quantum theory L_m's are given by

$$\boxed{L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}\cdot\alpha_n:}$$

For $m \neq 0$, α^μ_{m-n} and α^μ_n commute and normal ordering becomes redundant (we can move lowering operators to the right without any contributions from the commutators).

For $m=0$, normal ordering has a non-trivial effect.

Note that,

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{-n}\cdot\alpha_n = \frac{1}{2}\alpha_0\cdot\alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_n\cdot\alpha_{-n}$$

$$= \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\sum_{\mu}\eta^\mu{}_\mu\sum_{n=1}^{\infty} n$$

while,

$$\boxed{L_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty}:\alpha_{-n}\cdot\alpha_n: = \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n}$$

Thus in the process of normal ordering one throws away an undetermined constant $\frac{D}{2}\sum_{n=1}^{\infty} n$ (A word of caution: in fact the infinite sum can be regularized and computed leading to the result $\frac{D}{2}\times(-1/12)$ which is not infinite, nor is undetermined! However, in the "old covariant quantization" presented here, certain contributions to L_0 are missing. These missing terms would affect the value of the constant had they been included. In the absence of this extra piece of information, we regard the constant as undetermined and henceforth denote it by "$-a$"). Thus L_0 is associated with an undetermined constant "a". The convention is to define L_0 as the normal ordered expression given above and then take "a" into account by replacing L_0 by $(L_0 - a)$ while promoting classical expressions to quantum ones.

Now using the commutation relations for α^μ_m's one can compute $[L_m, L_n]$. The computation is not reproduced here, but the outcome is

$$\boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12}(m^3 - m)\,\delta_{m+n,0}}$$

This is the <u>Virasoro algebra</u> with a <u>central extension</u> term. The central extension is relevant only for $m=-n$. D which here corresponds to space-time dimension is called the <u>central charge</u>, in general denoted by c. For $m=0,\pm 1$, the central extension vanishes and furthermore, L_0, L_{+1}, L_{-1} form a closed subalgebra called the $SL(2,R)$ algebra.

The central extension arises as a quantum effect and does not exist at the classical level. If we ignore this piece for a while, then the rest of the algebra can be understood in very simple terms: We have seen that in open string theory L_m are conserved charges associated with conformal transformations, which have the infinitesimal form,

$$\delta\sigma^{\pm} = \sum_m \varepsilon_m e^{2\pi i m \sigma^{\pm}/\ell} = \varepsilon^{\pm}$$

More L_m is associated with the mode ε_m of the transformation. In analogy with ordinary quantum mechanics (where coordinate shifts $x \to x+\varepsilon$ are generated by the associated conserved momentum with representation $-i\partial/\partial x$) we can easily write a representation for L_m as an operator acting on functions of $\sigma^{\pm}$:

$$L_m = i\frac{\ell}{2\pi}\left(e^{2\pi i m\sigma^+/\ell}\partial_+ + e^{2\pi i m \sigma^-/\ell}\partial_-\right)$$

Hence, $\delta f(\sigma^+,\sigma^-) = \left(\frac{2\pi}{i\ell}\right)\sum_m \varepsilon_m L_m f(\sigma^+,\sigma^-)$

It is now easy to check that

$$[L_m, L_n] = (m-n)L_{m+n} \qquad \text{(classical)}$$

This is the classical Virasoro algebra associated with conformal transformations. At the quantum level it gets modified through the appearance of the central extension term. Later we relate this to the modification of the conformal transformation properties of the stress energy tensor.

<u>Remark</u>: The normalization of L_m is chosen as $+\frac{\ell}{2\pi}$ so that the Virasoro algebra has the standard form. Then a finite shift is implemented by

$$f(\sigma^{\pm}+\varepsilon^{\pm}) = e^{-i\frac{2\pi}{\ell}\sum_m \varepsilon_m L_m} f(\sigma^{\pm}) \equiv e^{-iQ_\epsilon} f(\sigma^{\pm})$$

This implies that when the theory is quantized and L_m (or Q_ϵ) are promoted to quantum operators, then under conformal transformations, a quantum state will transform as

$$|s'\rangle = e^{-iQ_\epsilon}|s\rangle \quad \Rightarrow \quad \delta_\epsilon |s\rangle = -iQ_\epsilon |s\rangle$$

and a quantum operator ϕ will transform as

$$\phi' = e^{-iQ_\epsilon}\phi\, e^{iQ_\epsilon} \quad \Rightarrow \quad \delta_\epsilon \phi = -i[Q_\epsilon, \phi],$$

where $Q_\epsilon = \frac{2\pi}{l}\sum_m \varepsilon_m L_m$ is the conserved Noether charge. We will need these formulae later and this discussion helps fix the sign conventions relative to Virasoro algebra.

Physical State conditions:

At the classical level, unphysical oscillation modes of the string are eliminated by the constraints $(L_m)_{\text{classical}} = 0$. In quantum theory classical quantities are replaced by the expectation values of the corresponding operators, hence the constraints become,

$$\langle\psi|(L_0 - a)|\psi\rangle = 0, \; \langle\psi|L_m|\psi\rangle = 0 \qquad \text{for } m \geq 1$$

or, in short,

$$\boxed{\langle\psi|(L_m - a\,\delta_{m,0})|\psi\rangle = 0}$$

where we have taken into account the normal ordering ambiguity in L_0. All states $|\psi\rangle$ that satisfy this constraint are retained as physical states and the rest are discarded. The hope is that this procedure will eliminate all negative norm states. One now has to find a way to implement the constraint systematically and easily characterize the physical states. The naive expectation of imposing the constraint through $(L_m - a\delta_{m,0})|\psi\rangle = 0$ for all m is not correct as this also implies $[L_m, L_{-m}]|\psi\rangle = 0$. But through the use of the Virasoro algebra this means $\left(2mL_0 + \frac{D}{12}(m^3 - m)\right)|\psi\rangle = \left(m(2a - \frac{D}{12}) + \frac{D}{12}m^3\right)|\psi\rangle = 0$ for all m, or essentially $|\psi\rangle = 0$.

However, the constraint can be consistently imposed through the weaker condition

$$\boxed{(L_m - a\,\delta_{m,0})|\psi\rangle = 0 \qquad \text{for } m \geq 0}$$

because, using the fact that $L_{-m} = L_m^{\dagger}$ and hence $\langle\psi|L_{-m} = (L_m|\psi\rangle)^{\dagger}$, one has $\langle\psi|(L_n - a\delta_{n,0})|\psi\rangle$ for all n. It is natural to impose the constraint in terms of L_m for $m \geq 0$ (rather than in terms of L_{-m}) since L_m lower the eigenvalue of the number operator $\mathbf{N}$ by m units.

Any state satisfying the above constraint is a physical state. It turns out that negative norm states are not part of the physical spectrum only when

$$\boxed{a = 1, \qquad D = 26}$$

In the "old covariant quantization" approach adopted here the proof of this statement is somewhat complicated and will not be presented here (light-cone and BRST quantization methods are more appropriate for this purpose).

Let us consider the implications of $(L_m - a\,\delta_{m,0})|\psi\rangle = 0$. Note that since $\alpha_0^\mu = p^\mu/\sqrt{T\pi}$ and $N = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m$, we have

$$\boxed{L_0 = \frac{1}{2\pi T}\, p^\mu p_\mu + N}$$

N has integer eigenvalues $0, 1, 2, \ldots$ and $M^2 = -p^\mu p_\mu$ is the mass squared operator. The L_0-condition then gives the mass formula for physical states,

$$(L_0 - a)|\text{phys}\rangle = 0 \quad \Rightarrow \quad \boxed{M^2 = -p^\mu p_\mu = 2\pi T\,(N - a)}$$

often it customary to use the notation $\alpha' = \frac{1}{2\pi T}$. Then the mass formula becomes

$$\boxed{\alpha' M^2 = N - a}$$

$\sqrt{2\pi T}$ is often taken to be of the order of the Planck mass and sets the energy scale of the theory.

The Open String Spectrum

The state with N eigenvalue $= 0$ is $|0, k\rangle$ for which $\alpha' M^2 \equiv -\alpha' k^2 = -a = -1$. Also $L_m|0, k\rangle = 0$ for $m \geq 1$.

Thus the first physical state in the theory of bosonic open strings is the tachyon with $M^2 = -1/\alpha'$. This is not to be regarded as a particle of imaginary mass! Rather, the presence of such a state signals an instability in the theory. This problem is circumvented in Superstring theory.

States with **N** eigenvalue = 1 are $\alpha^{\mu}_{-1}|0,k\rangle$, for $\mu = 0, 1, \ldots, 25$. In general, one can consider linear combinations of these basis states

$$|\psi_1\rangle = \sum_{\mu=0}^{25} \varepsilon_\mu \, \alpha^{\mu}_{-1} |0,k\rangle$$

where ε_μ could also depend on k^μ. The norm can be easily computed as

$$\langle\psi_1|\psi_1\rangle = \varepsilon_\mu \varepsilon^\mu = -(\varepsilon^0)^2 + (\varepsilon^1)^2 + (\varepsilon^2)^2 + \cdots + (\varepsilon^{25})^2$$

and could be positive, negative or zero, depending on the value of ε^0 relative to other components. The physical state conditions are

$$(L_0 - a)|\psi_1\rangle = 0 \quad \Rightarrow \quad \alpha' M^2 \equiv -\alpha' k^2 = 1 - a$$

$$L_1|\psi_1\rangle = 0 \quad \Rightarrow \quad \varepsilon_\mu k^\mu = 0$$

$$L_m|\psi_1\rangle = 0 \;\; (m \geq 2) \quad \Rightarrow \quad \text{no further conditions.}$$

To clarify the implications of these conditions in relation to the presence of $-ve$ norm states, note that by a spatial rotation one can always make the x^1 direction (say), parallel to $\vec{k}$ (the space part of k^μ). In this coordinate system,

$$k^\mu = \{k^0, k^1, 0, 0 \ldots, 0\}$$

and $$\varepsilon_\mu k^\mu = \varepsilon_0 k^0 + \varepsilon_1 k^1 = 0$$

It is obvious that 24 components of ε_μ i.e. $\varepsilon_2, \ldots \varepsilon_{25}$ are not constrained. Since their contribution to the norm is positive there is no problem with this. So for simplicity, we now focus on

$$\varepsilon_\mu = \{\varepsilon_0, \varepsilon_1, 0, \ldots, 0\}$$

which contains the undesired component ε_0. The situation now depends on the nature of k^2:

i) $a > 1 \Rightarrow k^2 = (a-1)/\alpha' > 0$: Since k^2 is space-like $(-k_0^2 + k_1^2 > 0)$, it is always possible to find a Lorentz transformation that sets k_0 to zero leading to $k^\mu = \{0, k^1, 0, \ldots 0\}$. Then $\varepsilon_\mu k^\mu = 0 \Rightarrow \varepsilon_1 = 0$ and the $-ve$ norm component ε_0 survives leading to an inconsistent theory. This rules out the choice $a > 1$.

ii) $a < 1 \Rightarrow k^2 < 0$: Since k^2 is time-like, one can find a Lorentz transformation that leads to $k^\mu = \{k^0, 0, 0, \ldots, 0\}$. Now, $\varepsilon_\mu k^\mu = 0 \Rightarrow \varepsilon_0 = 0$. Thus the negative norm state is eliminated and one ends up with a 25 component state $(\varepsilon_1, \ldots \varepsilon_{25})$ with $M^2 > 0$. Although naively this seems acceptable, it turns out that $a<1$ too leads to an inconsistent theory (This would correspond to a theory of massive vector fields which is non-renormalizable as a field theory).

iii) $a=0 \Rightarrow k^2=0$. In this case, k^2 is light-like, $k^\mu = \{k, \pm k, 0, \dots 0\}$ and $\varepsilon_\mu k^\mu = 0 \Rightarrow \varepsilon^0 = \pm\varepsilon^1$, or $\varepsilon^\mu = \{\varepsilon, \pm\varepsilon, 0, \dots\}$. In other words, ε^μ is proportional to k^μ. As a consequence, $\varepsilon_\mu \varepsilon^\mu = -(\varepsilon^0)^2 + (\varepsilon^1)^2 = 0$. Thus in this case of $a=1$, the harmfull (−ve norm) contribution of ε^0 cancels against ε^1 and we are left with a $M^2=0$ state with 24 components $\varepsilon^2, \varepsilon^3, \dots, \varepsilon^{25}$. This is how the constraints decouple negative norm states from the theory.

States with N eigenvalue = 2 are of the forms $\alpha^\mu_{-2}|0,k\rangle$ and $\alpha^\mu_{-1}\alpha^\nu_{-1}|0,k\rangle$ with $\alpha' M^2 = 1$ (for $a=1$). For $1/\sqrt{\alpha'}$ ~ Planck mass, these states (and the rest of the physical states) are highly massive and cannot be produced as external particles in scattering processes.

Summary:

$N=0$ $\quad T(k)|0,k\rangle \quad \alpha' M^2 = -\alpha' k^2 = -1 \quad$ (tachyon)

$N=1$ $\quad \varepsilon_\mu(k)\,\alpha^\mu_{-1}|0,k\rangle, \quad \left\{\begin{matrix}\alpha' M^2 = -\alpha' k^2 = 0 \\ \varepsilon_\mu k^\mu = 0\end{matrix}\right\} \quad$ (Massless vector boson)

$N=2$ $\quad \begin{matrix} c_\mu(k)\,\alpha^\mu_{-2}|0,k\rangle \\ e_{\mu\nu}(k)\,\alpha^\mu_{-1}\alpha^\nu_{-1}|0,k\rangle, \end{matrix} \quad \left\{\begin{matrix}\alpha' M^2 = 1 \\ \dots\end{matrix}\right\} \quad$ (very massive states)

<u>Remark</u>: To write down the most general linear combination of states, we have introduced the coefficients $T(k)$, $\varepsilon_\mu(k)$, etc. These are called the polarization tensors in analogy with quantum field theory. In fact, this is more than an analogy as these really turn out to correspond to QFT polarization tensors. For example, for the tachyon, let us introduce the Fourier transform

$$T(x) = \int d^D k \, e^{ik.x} \, T(k)$$

Then, the mass shell condition (the L_0 constraint) $\alpha' k^2 T(k) = T(k)$ implies $(\partial_\mu \partial^\mu + 1/\alpha') T(x) = 0$, which is the equation of motion for a free scalar field of mass $M^2 = -1/\alpha'$. Similarly, for the $N=1$ state,

$$A_\mu(x) = \int d^D k \, \varepsilon_\mu(k) \, e^{ik.x}$$

Then $k^2 \varepsilon_\mu = 0$ and $\varepsilon_\mu k^\mu = 0$ imply,

$$\partial_\mu \partial^\mu A_\nu = 0, \qquad \partial_\mu A^\mu = 0$$

This is the equation of motion for the photon field in the Lorenz gauge. Of course, at the level of non-interacting theories, these are merely kinematic statements and are not enough for identifying $\alpha_{-1}^\mu |0,k\rangle$ as the photon state in 26 dimensions. Hower even at the interactive level, string scattering amplitudes that are written in terms of polarization tensors have a low energy limit in which they are identified with the corresponding QFT amplitudes. This in fact is one way of deriving effective lowenergy field theories corresponding to string theories.

<u>Risidual gauge invariance</u>: In Lorenz gauge $\partial_\mu A^\mu = 0$, there is still a residual gauge invariance $A_\mu \to A_\mu + \partial_\mu \Lambda$ with $\partial^\mu \partial_\mu \Lambda = 0$. Then $\Lambda(x) = \int d^D k\, \lambda(k)\, e^{ik\cdot X}$ with $k^2 = 0$, and in terms of the polarization tensor $\varepsilon_\mu(k)$, the gauge transformation becomes $\varepsilon_\mu \to \varepsilon_\mu + \lambda k_\mu$.

In string theory this arises as follows: consider the state

$$|\delta\psi\rangle_\lambda = \lambda L_{-1}|0,k\rangle = \lambda k_\mu \alpha^\mu_{-1}|0,k\rangle .$$

Then,

$${}_\lambda\langle\delta\psi|\delta\psi\rangle_\lambda = 0 \qquad (\text{since } \lambda^2 k_\mu k^\mu = 0)$$

and

$$\langle\psi|\delta\psi\rangle_\lambda = 0 \qquad (\text{since } L_1|\psi\rangle = 0, \text{ for any physical state } |\psi\rangle).$$

Hence, $|\delta\psi\rangle_\lambda$ has zero norm and is orthogonal to all physical states. This means that in the calculation of any physical quantity one could equally well use $|\psi\rangle + |\delta\psi\rangle_\lambda$ instead of $|\psi\rangle$. The effect of this on the polarization tensor is the shift $\varepsilon_\mu \to \varepsilon_\mu + \lambda k_\mu$ as desired. In general, in string theory gauge symmetries are related to the presence of null states. Physical states are then defined as equivalent classes of states modulo such null states.