## Open String with Dirichlet Boundary Conditions (D-branes) :

Let us divide the set of D coordinates $X^\mu$ $(\mu = 0, 1, \ldots D-1)$ into two sets $X^a$ $(a = 0, 1, 2, \ldots p)$ and $X^i$ $(i = p+1, \ldots, D-1)$. On the $p+1$ coordinates $X^a$ we apply Neumann boundary conditions as discussed earlier. On the $D-p-1$ coordinates $X^i$ we apply Dirichlet boundary conditions,

$$\boxed{\partial_\tau X^i\Big|_{\sigma=0} = 0, \qquad \partial_\tau X^i\Big|_{\sigma=\bar{l}} = 0}$$

Starting with the general solution,

$$X^i = x^i + \frac{2}{lT} p^i \tau + \frac{2}{\sqrt{T}\, l} L^i \sigma + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \left( \frac{\alpha_n^i}{n} e^{-2\pi i n (\tau - \sigma)/l} + \frac{\tilde{\alpha}_n^i}{n} e^{-2\pi i n (\tau + \sigma)/l} \right)$$

the boundary condition $\partial_\tau X^i\Big|_{\sigma=0} = 0$ leads to

$$\boxed{p^i = 0, \qquad \alpha_n^i = -\tilde{\alpha}_n^i}$$

The boundary condition $\partial_\tau X^i\Big|_{\sigma=\bar{l}} = 0$ is then satisfied provided,

$$\boxed{\bar{l} = l/2}$$

as in the case of Neumann boundary conditions. The solution for $X^i(\sigma, \tau)$ then becomes,

$$X^i(\sigma,\tau) = x^i + \frac{2L^i}{\ell\sqrt{T}}\sigma + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{\alpha_n^i}{n}\left(e^{-2\pi i n(\tau-\sigma)/\ell} - e^{-2\pi i n(\tau+\sigma)/\ell}\right)$$

$$= x^i + \frac{2L^i}{\ell\sqrt{T}}\sigma - \frac{1}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{\alpha_n^i}{n}\sin(2\pi n\sigma/\ell)\, e^{-2\pi i n\tau/\ell}$$

As **before**, we compute,

$$\boxed{\left.\begin{aligned} \partial_+ X^i &= -\frac{\sqrt{\pi}}{\ell\sqrt{T}}\sum_{n=-\infty}^{\infty}\alpha_n^i\, e^{-2\pi i n(\tau+\sigma)/\ell} \\ \partial_- X^i &= \frac{\sqrt{\pi}}{\ell\sqrt{T}}\sum_{n=-\infty}^{\infty}\alpha_n^i\, e^{-2\pi i n(\tau-\sigma)/\ell} \end{aligned}\right\} \quad \alpha_0^i = -\frac{L^i}{\sqrt{\pi}}}$$

Note again that the solution is expressed in terms of functions $\exp(2\pi i n\sigma/\ell)$ that form a complete orthogonal set over $0\leqslant\sigma\leqslant\ell$ (or $-\frac{\ell}{2}\leqslant\sigma\leqslant\ell/2$), which is double the length of the open string. Verify that

$$X^i(\sigma) \xrightarrow{\sigma\to-\sigma} X^i(-\sigma) = -X^i(\sigma) + 2x^i$$

$$\Rightarrow$$

$$\partial_- X^i(\sigma) \xrightarrow{\sigma\to-\sigma} -\partial_+ X^i(\sigma)$$

The stress-energy tensor is

$$T_{\pm\pm} = \partial_\pm X^\mu \partial_\pm X_\mu = \partial_\pm X^a \partial_\pm X_a + \partial_\pm X^i \partial_\pm X_i$$

Invariance under conformal transformations leads to conserved charges

$$Q \sim \int_0^{\bar{\ell}} d\sigma \left( T_{++}\varepsilon^+ + T_{--}\varepsilon^- \right)$$

provided $\left. \left( T_{++}\varepsilon^+ - T_{--}\varepsilon^- \right) \right|_{\substack{\sigma=0\\ \sigma=\bar{\ell}}} = 0$

$\Rightarrow$

$$\left. \left( \partial_\sigma X^i \partial_\sigma X_i + \partial_\tau X^a \partial_\tau X_a \right)\left( \varepsilon^+ - \varepsilon^- \right) \right|_{\substack{\sigma=0\\ \sigma=\bar{\ell}}} = 0 \quad \text{or} \quad \left. \varepsilon^+ - \varepsilon^- \right|_{\substack{\sigma=0\\ \sigma=\ell}} = 0$$

Thus, as in the case of pure Neumann boundary conditions,

$$\varepsilon^+ = \sum_k \varepsilon_k e^{2\pi i k(\tau+\sigma)/\ell}, \qquad \varepsilon^- = \sum_k \varepsilon_k e^{2\pi i k(\tau-\sigma)/\ell}$$

Corresponding to transformations generated by $\varepsilon_k$ and using

$$T_{\pm\pm} = \frac{\pi}{\ell^2 T} \sum_{n,m} \alpha_n^\mu \alpha_{m,\mu} e^{-2\pi i(n+m)(\tau\pm\sigma)/\ell}$$

we obtain the conserved charges,

$$Q_\varepsilon = \frac{T\ell}{2\pi} \int_0^{\ell/2} d\sigma \left( T_{++}(\tau,\sigma)\varepsilon^+(\tau,\sigma) + T_{--}(\tau,\sigma)\varepsilon^-(\tau,\sigma) \right)$$

$$= \frac{T\ell}{2\pi} \left( \int_0^{\ell/2} d\sigma\, T_{++}(\tau,\sigma)\varepsilon^+(\tau,\sigma) + \int_{-\ell/2}^{0} d\sigma\, \underbrace{T_{--}(\tau,-\sigma)\varepsilon^-(\tau,-\sigma)}_{T_{++}(\tau,\sigma)\varepsilon^+(\tau,\sigma)} \right)$$

$$= \frac{T\ell}{2\pi} \int_{-\ell/2}^{\ell/2} d\sigma\, T_{++}\varepsilon^+(\tau,\sigma)$$

$$L_k = \frac{T\ell}{2\pi} \int_{-\ell/2}^{\ell/2} d\sigma \left( \frac{\pi}{\ell^2 T} \right) \sum_{n,m} \alpha_n^\mu \alpha_{m,\mu} e^{-2\pi i(n+m)(\tau+\sigma)/\ell} e^{2\pi i k(\tau+\sigma)/\ell}$$

Finally, using $\int_{-\ell/2}^{\ell/2} d\sigma\, e^{2\pi i(k-n-m)\sigma/\ell} = \ell\,\delta_{m+n-k,0}$ gives,

$$L_k = \frac{1}{2}\sum_m \alpha^\mu_{k-m}\alpha_{m,\mu} = \frac{1}{2}\sum_m \left(\alpha^a_{k-m}\alpha_{m,a} + \alpha^i_{k-m}\alpha_{m,i}\right).$$

In particular,

$$L_0 = \frac{1}{2}\alpha^a_0\alpha_{0,a} + \frac{1}{2}\alpha^i_0\alpha_{0,i} + \frac{1}{2}\sum_{m\neq 0}\alpha^\mu_m\alpha_{m,\mu}.$$

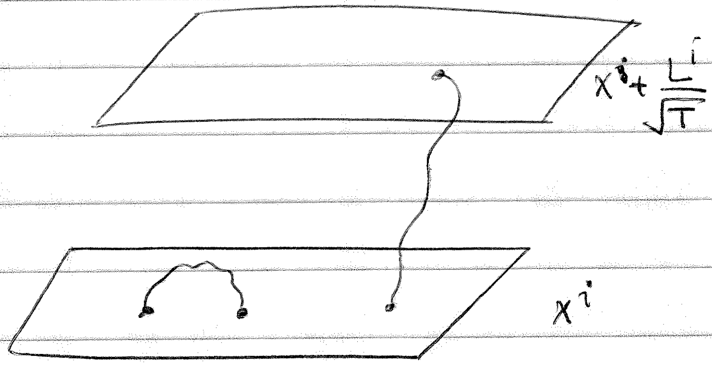
Using $\alpha^i_0 = \frac{L^i}{\sqrt{\pi}}$, $\alpha^a_0 = \frac{p^a}{\sqrt{\pi T}}$, the constraint $L_0 = 0$ gives,

$$\boxed{M^2 = -p^a p_a = T L^i L_i + \pi T \sum_{m\neq 0}\left(\alpha^a_{-m}\alpha_{m,a} + \alpha^i_{-m}\alpha_{m,i}\right)}$$

The presence of $L^i$ in $X^i$ indicates that the two end points of the open string could be at different space points:

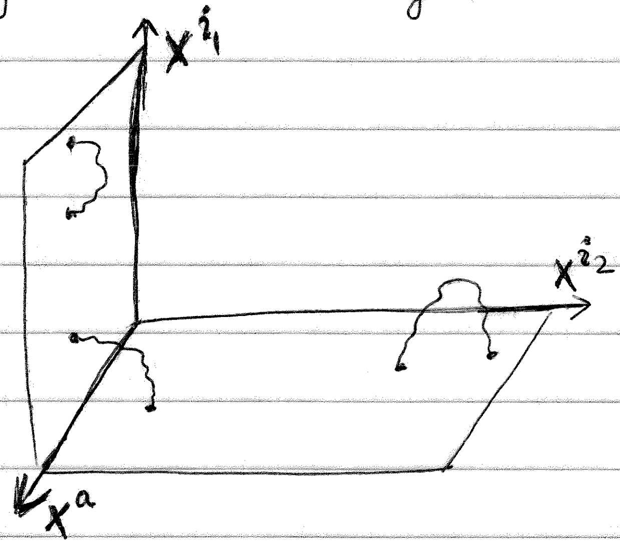
$$X^i(\sigma = \ell/2) - X^i(\sigma = 0) = \frac{L^i}{\sqrt{T}}$$

This contributes to the string mass at the classical level. Since the string end points can only move in the $X^a$ directions it only makes sense to define mass in terms of $p^a$.



## Mixed (ND) Boundary Conditions & Intersecting Branes

For open strings, the boundary term $\partial_\sigma X^\mu \delta X_\mu \big|_{\sigma=0,\ \sigma=\ell}$ that arose from the variation of the action should vanish at each end point separately. This allows for the possibility of imposing Neumann boundary conditions at one end and Dirichlet boundary conditions at the other end of the open string. This type of situation is realized by intersecting D-branes. For concreteness consider the following set up: Break up the space-time coordinates $X^\mu$ $(\mu = 0, \dots D-1)$ into 4 sets $X^{\hat{i}_1}$, $X^{\hat{i}_2}$, $X^a$ and $X^i$. The $1^{st}$ D-brane has a worldvolume along $X^a$ and $X^{\hat{i}_2}$ with $X^{\hat{i}_1}$ and $X^i$ as transverse direction. This is associated with open strings with N boundary conditions along $X^a$ and $X^{\hat{i}_2}$ and D boundary conditions along $X^{\hat{i}_1}$ and $X^i$. The $2^{nd}$ D-brane has a worldvolume along $X^a$ and $X^{\hat{i}_1}$ with $X^{\hat{i}_2}$ and $X^i$ as transverse directions. The corresponding open string boundary conditions are of N type along $X^a$ and $X^{\hat{i}_1}$ and of D type along $X^{\hat{i}_2}$ and $X^i$.




We have already studied such open strings and there is nothing new in this. However now we also have a third set of open strings that start from one of the branes and end on the other one. These strings will have the standard D boundary conditions along the common transverse directions $X^i$ and N boundary conditions along the common world volume directions $X^a$. However, along $X^{i_1}$ and $X^{i_2}$ they will satisfy mixed boundary conditions: Consider a string begining on the $1^{st}$ brane and ending on the second brane. Along $X^{i_1}$ it satisfies DN boundary condition:

$$\text{D}: \quad \partial_\tau X^{i_1}\Big|_{\sigma=0} = 0 \quad , \qquad \text{N}: \quad \partial_\sigma X^{i_1}\Big|_{\sigma=\bar{l}} = 0$$

Along $X^{i_2}$ is satisfies ND boundary conditions:

$$\text{N}: \quad \partial_\sigma X^{i_2}\Big|_{\sigma=0} = 0 \quad , \qquad \text{D}: \quad \partial_\tau X^{i_2}\Big|_{\sigma=\bar{l}} = 0$$

Starting from the general solution,

$$X^{i_1} = x^{i_1} + \frac{2}{lT} p^{i_1}\tau + \frac{2}{l\sqrt{T}} L^{i_1}\sigma + \frac{i}{\sqrt{4\pi T}} \sum_{r\neq 0} \left( \frac{\alpha_r^{i_1}}{r} e^{-2\pi i r(\tau-\sigma)/l} + \frac{\bar{\alpha}_r^{i_1}}{r} e^{-2\pi i r\frac{(\tau+\sigma)}{l}} \right)$$

$$\partial_\tau X^{i_1}\Big|_{\sigma=0} = 0 \quad \Rightarrow \quad p^{i_1} = 0, \quad \alpha_r^{i_1} = -\bar{\alpha}_r^{i_1}$$

with this, $\displaystyle \partial_\sigma X^{i_1} = \frac{2L^{i_1}}{l\sqrt{T}} - \frac{2\sqrt{\pi}}{l\sqrt{T}} \sum_{r\neq 0} \alpha_r^{i_1} \cos(2\pi r\sigma/l)\, e^{-2\pi i r\tau/l}$

Then $\displaystyle \partial_\sigma X^{i_1}\Big|_{\sigma=\bar{l}} = 0 \Rightarrow L^{i_1} = 0, \quad \bar{l} = l/2, \quad r = n + \frac{1}{2}$

($n$ = integer).

Thus we get,

$$\boxed{X^{i_1}(\sigma,\tau) = x^{i_1} - \frac{1}{\sqrt{\pi T}} \sum_{r=n+\frac{1}{2}} \frac{\alpha_r^{i_1}}{r} \sin(2\pi r\sigma/\ell)\, e^{-2\pi i r\tau/\ell}}$$

The new feature of this expansion is that the oscillators are now half-odd-integer moded. Note that the end $\sigma=0$ is fixed at $X^i = x^i$ while the $\sigma=\ell/2$ end oscillates around this value, with the average value still being $x^i$.

As for the $X^{i_2}$ directions,

$$\partial_\sigma X^{i_2}\Big|_{\sigma=0} = 0 \Rightarrow \quad L^{i_2}=0, \quad \bar{\alpha}_r^{i_2} = \alpha_r^{i_2}$$

Then,

$$\partial_\tau X^{i_2}\Big|_{\sigma=\bar{\ell}} = 0 \Rightarrow \frac{2p^{i_2}}{\ell T} + \frac{2\sqrt{\pi}}{\ell\sqrt{T}} \sum_r \alpha_r^{i_2} \cos(2\pi r\sigma/\ell)\, e^{-2\pi i r\tau}\Big|_{\sigma=\bar{\ell}} = 0$$

$$\Rightarrow \quad p^{i_2} = 0, \quad \bar{\ell} = \ell/2, \quad r = n+1/2,\ n\in\mathbb{Z}$$

So that,

$$\boxed{X^{i_2}(\sigma,\tau) = x^{i_2} + \frac{i}{\sqrt{T\pi}} \sum_{r=n+\frac{1}{2}} \frac{\alpha_r^{i_2}}{r} \cos(2\pi r\sigma/\ell)\, e^{-2\pi i r\tau/\ell}}$$

Note that the structures of $X^{i_1}$ & $X^{i_2}$ are essentially related by $\sigma \to \ell/2 - \sigma$ with a redefinition of $\alpha_r^{i_{1/2}}$.

The string does not have center of mass motion along $X^{i_1}$ and $X^{i_2}$ ($p^{i_1}=0$, $p^{i_2}=0$) due to the ND boundary conditions. In this sense ND & DN strings are **trapped** at the intersection region. A special case is when there are no $X^{i_2}$ directions. In this case, the 1st brane is entirely inside the 2nd brane.

To compute the $L_n$'s note that

$$\partial_+ X^{i_1} = \frac{\sqrt{\pi}}{\ell\sqrt{T}} \sum_r \alpha_r^{i_1} e^{-2\pi i r(\tau+\sigma)/\ell}, \quad \partial_- X^{i_1} = \frac{-\sqrt{\pi}}{\ell\sqrt{T}} \sum_r \alpha_r^{i_1} e^{-2\pi i r(\tau-\sigma)/\ell}$$

$$\partial_+ X^{i_2} = \frac{\sqrt{\pi}}{\ell\sqrt{T}} \sum_r \alpha_r^{i_2} e^{-2\pi i r(\tau+\sigma)/\ell}, \quad \partial_- X^{i_2} = \frac{\sqrt{\pi}}{\ell\sqrt{T}} \sum_r \alpha_r^{i_2} e^{-2\pi i r(\tau-\sigma)/\ell}$$

The contributions of these to $T_{++}$ & $T_{--}$ are

$$(T_{++})_{ND} = \frac{\pi}{\ell^2 T} \sum_{r,s} (\alpha_r \cdot \alpha_s)_{ND}\, e^{-2\pi i (r+s)(\tau+\sigma)/\ell}$$

$$(T_{--})_{ND} = \frac{\pi}{\ell^2 T} \sum_{r,s} (\alpha_r \cdot \alpha_s)_{ND}\, e^{-2\pi i (r+s)(\tau-\sigma)/\ell}$$

The charge conservation condition $\left.(T_{++}\varepsilon^+ - T_{--}\varepsilon^-)\right|_{\substack{\sigma=0\\ \sigma=\ell}} = 0$ leads to

$$\left.(\varepsilon^+ - \varepsilon^-)\right|_{\substack{\sigma=0\\ \sigma=\ell/2}} = 0 \Rightarrow \varepsilon^\pm = \sum_k \varepsilon_k e^{2\pi i k(\tau\pm\sigma)/\ell}.$$

Using the doubling trick to combine $T_{++}$ & $T_{--}$ into a single $T_{++}$ defined over $-\ell/2 \le \sigma \le \ell/2$, the conserved charge corresponding to the $\varepsilon_k$ mode is obtained as

$$L_k = \frac{T\ell}{2\pi} \int_{-\ell/2}^{\ell/2} d\sigma \, T_{++} \, e^{2\pi i k(\tau+\sigma)/\ell}$$

$$\boxed{\begin{aligned} L_k = \frac{1}{2} \sum_{s \in \mathbb{Z}+1/2} \left( \alpha^{i_1}_{k-s} \alpha_{s,i_1} + \alpha^{i_2}_{k-s} \alpha_{s,i_2} \right) \\ + \frac{1}{2} \sum_n \left( \alpha^a_{k-n} \alpha_{n,a} + \alpha^i_{k-n} \alpha_{n,i} \right) \end{aligned}}$$

we have included contributions from NN and DD coordinates.

$$\alpha^a_0 = \frac{p^a}{\sqrt{\pi T}}, \qquad \alpha^i_0 = \frac{L^i}{\sqrt{\pi}}$$