Classical Dynamics of Strings

So far, we have studied aspects of the Polyakov action related to the worldsheet metric $h_{\alpha\beta}$. Now we consider the dynamics of the embedding functions $X^\mu(\sigma,\tau)$ by evaluating their equations of motion using the least action principle. Under a variation $X^\mu \to X^\mu + \delta X^\mu$ in the functional form of X^μ, setting $\delta S_p = 0$ gives the equation of motion for X^μ:

$$S_p = -\frac{T}{2}\int d^2\sigma \left(\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right)$$

$\Rightarrow$

$$\delta S_p = -T\int d^2\sigma\left(\eta^{\alpha\beta}\partial_\alpha(\delta X^\mu)\partial_\beta X_\mu\right) = -T\int d^2\sigma\, \partial_\alpha\left(\eta^{\alpha\beta}\delta X^\mu \partial_\beta X_\mu\right)$$
$$+ T\int d^2\sigma\, \delta X^\mu\left(\partial^\alpha\partial_\alpha X_\mu\right)$$

$$= T\int d^2\sigma\,(\partial^\alpha\partial_\alpha X^\mu)\delta X_\mu + T\int_{-\infty}^{+\infty} d\tau \int_0^{\bar{l}} d\sigma\left[\partial_\tau(\delta X^\mu \partial_\tau X_\mu) - \partial_\sigma(\delta X^\mu\partial_\sigma X_\mu)\right]$$

$$= T\int d^2\sigma(\partial^\alpha\partial_\alpha X^\mu)\delta X_\mu + T\int_0^{\bar{l}} \delta X^\mu \partial_\tau X_\mu\Big|_{\tau=-\infty}^{\tau=+\infty} d\sigma - T\int_{-\infty}^{\infty} d\tau\,(\delta X^\mu\partial_\sigma X_\mu)\Big|_{\sigma=0}^{\sigma=\bar{l}}$$

As in ordinary field theory, we can take the field variation δX^μ to vanish at infinity (in this case $\tau = \pm\infty$). Hence δS_p gives us:

$$\boxed{\begin{array}{ll} \partial^\alpha\partial_\alpha X^\mu = 0 & \text{(Eqn. of motion)} \\ \partial_\sigma X^\mu \delta X_\mu\Big|_{\sigma=\bar{l}} - \partial_\sigma X^\mu \delta X_\mu\Big|_{\sigma=0} = 0 & \text{(Boundary conditions)} \end{array}}$$

Note that in ordinary 3+1 dimensional field theory, the spatial boundary is at infinity and hence the corresponding boundary conditions can be ignored. But on the string worldsheet, the spatial boundary is at finite distance ($\sigma = 0, \bar{l}$) and its contribution to δS_p should be set to zero to extremize the action. Thus in this case, the boundary conditions always supplement the equation of motion.

Boundary Conditions :

There are various ways of satisfying the boundary conditions,

$$\boxed{\partial_\sigma X^\mu \delta X_\mu \Big|_{\sigma=\bar{l}} - \partial_\sigma X^\mu \delta X_\mu \Big|_{\sigma=0} = 0} \qquad (BC)$$

each leading to a different type of string :

① Closed strings : a closed string is a closed loop of circumference $\bar{l}$

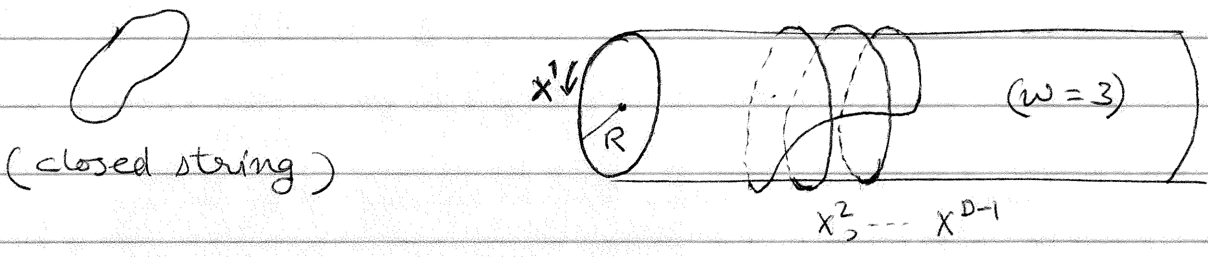
a) closed strings in non-compact flat space : Since $\sigma = 0$ and $\sigma = \bar{l}$ are the same point on the string, they also correspond to the same point in space-time ;

$$\boxed{X^\mu(\sigma + \bar{l}) = X^\mu(\sigma)}$$

δX^μ is a variation over the space of periodic functions and hence is periodic itself, $\delta X^\mu(\sigma+\bar{l}) = \delta X^\mu(\sigma)$. Thus the two terms in equation (BC) cancel and the boundary condition is satisfied.

b) Flat space compactified on a circle: Let one of the space dimensions, say X^1 consist of a circle of radius R, $\quad X^1 \simeq X^1 + 2\pi R$. A closed string can wind around this circle w time before closing onto itself,

$$\boxed{X^1(\sigma+\bar{l}) = X^1(\sigma) + 2\pi R w}$$



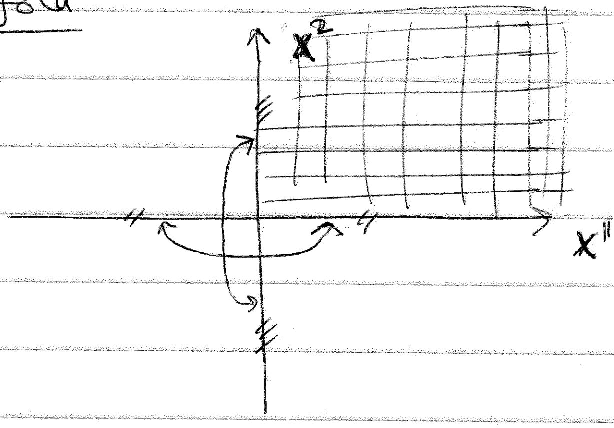
This is the simplest example of a compact space.

c) A simple orbifold: Since (BC) is quadratic in X^μ, it is also satisfied by

$$\boxed{X^1(\sigma+\bar{l}) = - X^1(\sigma)}$$

As σ and $\sigma+\bar{l}$ are the same point on the closed string, this boundary condition makes sense only if X^1 and $-X^1$ are the same point in space. This corresponds to a space where the X^1 axis is folded on itself identifying any value X^1 with $-X^1$ $(\mathbb{R}/\mathbb{Z}_2)$

Such boundary conditions can be imposed along several coordinates. The resulting space is a simple example of an orbifold



(2) Open strings : An open string is parametrized by $0 \leq \sigma \leq \bar{l}$ with its two ends having space-time coordinates $X^\mu(\sigma=0)$ and $X^\mu(\sigma=\bar{l})$. The two terms in equation (BC) are now evaluated at different space-time points (for every value of τ) and hence cannot be made to cancel against each other as this would imply non-local effects of the type that contradict causality. To avoid this problem, one sets each term in (BC) to zero independently. The simplest way of acheiving this is through the open string boundary conditions,

$$\boxed{\partial_\sigma X^\mu\Big|_{\sigma=0} = 0, \quad \partial_\sigma X^\mu\Big|_{\sigma=\bar{l}} = 0 \quad \text{(Neumann boundary condition)}}$$

Open strings satisfying this boundary condition can be found anywhere in the D dimensional space time.

③ Open strings on D-branes : Another way of satisfying Eqn (BC) for open strings is to demand that the variation δX is zero at the end points of the string leading to the so called Dirichlet boundary conditions. In general, one can impose a mixture of Neumann and Dirichlet boundary conditions:

$$\boxed{\begin{aligned} &\partial_\alpha X^m \Big|_{\substack{\sigma=0\\ \sigma=\bar{l}}} = 0 \quad \text{for } m=0,1,\dots p \text{ (Neumann)} \\ &\delta X^i \Big|_{\substack{\sigma=0\\ \sigma=\bar{l}}} = 0 \quad \text{for } i=p+1,\dots D-1 \text{ (Dirichlet)} \end{aligned}}$$

Physically, Dirichlet boundary condition means that the X^i coordinates of the end points of the string are fixed (while the end points can move in the X^m directions). Now, fixing the values of the $D-p-1$ coordinates X^i defines a $p+1$ dimensional hyper-surface in the D-dimensional space-time. All points within this hyper-surface have the same X^i coordinates. Changing the values of X^i corresponds to moving away from this hyper-surface. The Dirichlet boundary condition describes open strings the ends of which remain confined to this hypersurface at all times. Such a hyper-surface (which extends along p space dimensions and 1 time dimension), together with open strings with end points confined to it, is called a Dirichlet **p-brane**, or in short, a Dp-brane.

A p-brane is a generalization of a membrane ($\equiv$ 2-brane) from 2 space dimensions to p space dimensions. In time, a p-brane sweeps a p+1 dimensional space time, called the world volume of the p-brane, in analogy with the worldline of a particle ($\equiv$ 0-brane)

Remarks on D-branes

(i) Although we have looked at D-branes that are parallel to some of the coordinate axes (X^m in this case), one can also consider configurations at angles.

(ii) In general, one can also consider "curved branes" as opposed to the flat ones above. However, in a flat space-time these are generically unstable (open strings fly off in straight lines while the brane curves, due to conservation of linear momentum).

(iii) A special realization of $\delta X^i \Big|_{\sigma=0,\bar{\ell}} = 0$ is provided by open string end points whose X^i coordinates do not change in τ:

$$\boxed{\partial_\tau X^i \Big|_{\substack{\sigma=0 \\ \sigma=\ell}} = 0 \quad \Rightarrow \quad \delta X^i \Big|_{\substack{\sigma=0 \\ \sigma=\bar{\ell}}} = 0}$$

(iv) While $\partial_\tau X^i\big|_{\sigma=0,\bar{l}} = 0 \Rightarrow \delta X^i\big|_{\sigma=0,\bar{l}} = 0$, the converse is not necessarily true. For example, the D-brane can have a fixed velocity along the X^i, $\partial_\tau X^i\big|_{\sigma=0,\bar{l}} = v^i$. As long as one considers X^i with a fixed τ-dependence, the functional variation $\delta X^i\big|_{\sigma=0,\bar{l}}$ will still be zero (a fixed function cannot be varied)

(v) Though the end points of the open string are confined to the hyper surface, other points on the string are not necessarily confined to this surface.

(vi) The X^i coordinates of the $\sigma=0$ and $\sigma=\bar{l}$ end can be fixed at different values. Then the end points of the string are on different branes. One also has open strings satisfying N boundary conditions at one end and D boundary conditions at the other end. These correspond to intersecting branes. The open string with pure Neumann boundary conditions correspond to the D9-brane. A brane with Dirichlet boundary condition along the time direction is called an "instantonic" Dbrane, these are configurations localized in time.

Translational Invariance and Momentum Conservation

The action S_P is invariant under constant translations of X^μ: $X^\mu \to X^\mu + a^\mu$, with σ^α unchanged. Noether's theorem then leads to the existence of D conserved currents

$$\pi_\mu^\alpha = \frac{\partial \mathcal{L}_P}{\partial(\partial_\alpha X^\mu)} = -T\partial^\alpha X_\mu$$

Current conservation coincides with the eqn. of motion for X^μ: $\partial_\alpha \pi_\mu^\alpha = 0 \Rightarrow \partial_\alpha \partial^\alpha X^\mu = 0$. The associated conserved charges are the D-dimensional space-time momenta P_μ:

$$\boxed{P_\mu = \int_0^{\bar{\ell}} d\sigma\, \pi_\mu^0 = T\int_0^{\bar{\ell}} d\sigma\, \partial_\tau X_\mu .}$$

$$\frac{\partial P^\mu}{\partial\tau} = 0 \Rightarrow \pi_1^\mu\Big|_{\sigma=0}^{\sigma=\bar{\ell}} = \partial_\sigma X^\mu\Big|_{\sigma=\bar{\ell}} - \partial_\sigma X^\mu\Big|_{\sigma=0} = 0$$

The boundary condition required by momentum conservation is automatically satisfied by closed strings as well as open string Neumann boundary conditions. However it is not consistent with open string Dirichlet boundary conditions $\left(\delta X^i\big|_{\sigma=0,\bar{\ell}} = 0 \text{ or } \partial_\tau X^i\big|_{\sigma=0,\bar{\ell}} = 0\right)$.

Hence in this case the momenta P_i $(i = p+1, \ldots D-1)$ are not conserved. The reason is that in the presence of a D-brane localized at a certain position along x^i, the system is no longer translationally invariant along these directions.

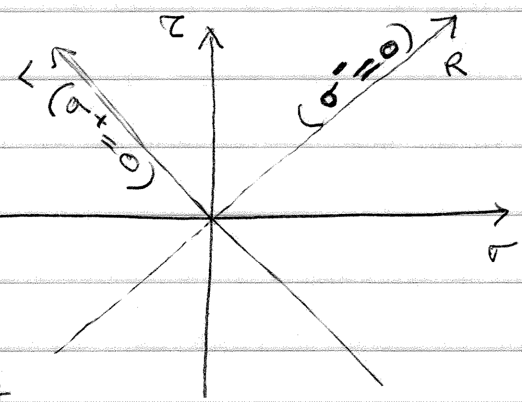
Equation of Motion and its Solutions

We have derived the equation of motion

$$\boxed{\partial^\alpha \partial_\alpha X^\mu = 0 \Rightarrow (\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0 \Rightarrow \partial_+ \partial_- X^\mu = 0}$$

This is the wave equation in 1+1 dimensions. The form $\partial_+ \partial_- X^\mu = 0$ implies that the solution is of the form

$$\boxed{X^\mu(\sigma,\tau) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)}$$



X_L^μ and X_R^μ denote waves that travel to the left and right in the $(\sigma-\tau)$ diagram. On these solutions we have to apply the periodic (in σ), Neumann, or Dirichlet boundary conditions, as the case may be. We then impose $T_{\alpha\beta} = 0$ to insure that X^μ describe physical motions of the string. A general solution consistent with the boundary conditions has the form:

$$X_L^\mu = x_L^\mu + \frac{1}{\ell T} p_L^\mu (\tau+\sigma) + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{\bar{\alpha}_n^\mu}{n} e^{-2\pi i n(\tau+\sigma)/\ell}$$

$$X_R^\mu = x_R^\mu + \frac{1}{\ell T} p_R^\mu (\tau-\sigma) + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-2\pi i n(\tau-\sigma)/\ell}$$

so that,

$$\text{(S)}\quad \boxed{X^\mu = x^\mu + \frac{1}{\bar{l}T} p^\mu \tau + \frac{1}{\bar{l}T} w^\mu \sigma + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \left(\frac{\alpha_n^\mu}{n} e^{-2\pi i n(\tau-\sigma)/l} + \frac{\bar{\alpha}_n^\mu}{n} e^{-2\pi i n(\tau+\sigma)/l} \right)}$$

$$p^\mu = p_L^\mu + p_R^\mu \,, \qquad w^\mu = p_L^\mu - p_R^\mu$$

$\bar{l}$ is the parametric length of the string, $0 \leq \sigma \leq \bar{l}$. l is a parameter related to $\bar{l}$ and the relation is determined by the boundary conditions. The solution is written in terms of functions $e^{2\pi i n \sigma/l}$ that are orthogonal and form a complete set over $0 \leq \sigma \leq l$:

$$\boxed{\begin{aligned} \frac{1}{l}\sum_{n=-\infty}^{\infty} e^{2i\pi n(\sigma-\sigma')/l} &= \delta(\sigma-\sigma') \\ \frac{1}{l}\int_0^l d\sigma\, e^{2\pi i(n-m)\sigma/l} &= \delta_{n,m} \end{aligned}}$$

We will see below that for closed strings, $\bar{l} = l$ and for open strings, $\bar{l} = l/2$. For both types of strings, the conserved space-time momentum is given by

$$P^\mu = T\int_0^{\bar{l}} d\sigma\, \partial_\tau X^\mu = p^\mu$$

The average value of the X^μ coordinate of the string is (at $\tau=0$):

$$\bar{X}^\mu = \frac{1}{\bar{l}} \int_0^{\bar{l}} d\sigma\, X^\mu\big|_{\tau=0} = x^\mu$$

In the solution for X^μ, the factors of T are chosen such that X^μ and x^μ have dimensions of length, p^μ (and w^μ) have dimensions of momentum (or $(\text{length})^{-1}$) and α_n^μ are dimensionless.

The reality of X^μ implies that

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu \,, \qquad (\bar{\alpha}_n^\mu)^\dagger = \bar{\alpha}_{-n}^\mu$$

Oscillator Expansion of Closed String Solution

The solution (S) satisfies the closed string boundary condition in flat non-compact space,

$$X^\mu(\sigma) = X^\mu(\sigma + \bar{l})$$

provided $l = \bar{l}$ and $w^\mu = p_L^\mu - p_R^\mu = 0$. Then $p_L^\mu = p_R^\mu = \frac{p^\mu}{2}$ and

$$X^\mu = x^\mu + \frac{1}{lT} p^\mu \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \left(\frac{\alpha_n^\mu}{n} e^{-2\pi i n(\tau-\sigma)/l} + \frac{\bar{\alpha}_n^\mu}{n} e^{-2\pi i n(\tau+\sigma)/l} \right)$$

$$X_L^\mu = x_L^\mu + \frac{1}{2lT} p^\mu(\tau+\sigma) + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{\bar{\alpha}_n^\mu}{n} e^{-2\pi i n(\tau+\sigma)/l}$$

$$X_R^\mu = x_R^\mu + \frac{1}{2lT} p^\mu(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-2\pi i n(\tau-\sigma)/l}$$

$$\partial_+ X^\mu = \partial_+ X_L^\mu = \frac{\sqrt{\pi}}{\ell\sqrt{T}} \sum_n \bar{\alpha}_n^\mu \, e^{-2\pi i n(\tau+\sigma)/\ell}$$

$$\partial_- X^\mu = \partial_- X_R^\mu = \frac{\sqrt{\pi}}{\ell\sqrt{T}} \sum_n \alpha_n^\mu \, e^{-2\pi i n(\tau-\sigma)/\ell}$$

where, we have defined,

$$\bar{\alpha}_0^\mu = \frac{1}{\sqrt{\pi T}} p_L^\mu = \frac{p^\mu}{\sqrt{4\pi T}}, \qquad \alpha_0^\mu = \frac{1}{\sqrt{\pi T}} p_R^\mu = \frac{p^\mu}{\sqrt{4\pi T}}$$

Let us now consider the stress energy tensor $T_{\alpha\beta}$ and the associated conserved charges, the L_n's (we use the notation $a^\mu b_\mu = a\cdot b$):

$$T_{++} = \partial_+ X\cdot\partial_+ X = \frac{\pi}{\ell^2 T}\sum_{m,n} \bar{\alpha}_m\cdot\bar{\alpha}_n \, e^{-2\pi i(n+m)(\tau+\sigma)/\ell}$$

$$T_{--} = \partial_- X\cdot\partial_- X = \frac{\pi}{\ell^2 T}\sum_{m,n} \alpha_m\cdot\alpha_n \, e^{-2\pi i(n+m)(\tau-\sigma)/\ell}$$

As discussed earlier, corresponding to conformal transformations $\delta\sigma^\pm = \varepsilon^\pm(\sigma^\pm)$ one gets conserved charges

$$Q_+ \sim \int_0^\ell d\sigma \, T_{++}\varepsilon^+, \qquad Q_- \sim \int_0^\ell d\sigma \, T_{--}\varepsilon^-$$

The requirement $\frac{\partial}{\partial\tau} Q_\pm = 0$ implies $\varepsilon^\pm(\sigma+\ell) = \varepsilon^\pm(\sigma)$.
Then,

$$\varepsilon^+(\sigma^+) = \sum_{k=-\infty}^{\infty} \varepsilon_k^+ \, e^{2\pi i k(\tau+\sigma)/\ell}, \qquad \varepsilon^-(\sigma^-) = \sum_{k=-\infty}^{\infty} \varepsilon_k^- \, e^{2\pi i k(\tau-\sigma)/\ell}$$

Thus we have a conserved charge corresponding to each mode of the transformation parameter. For the k^{th} mode we define the conserved quantities

$$L_k = \frac{T\ell}{2\pi}\int_0^\ell d\sigma\, T_{--}\, e^{2\pi i k(\tau-\sigma)/\ell}$$

$$\bar{L}_k = \frac{T\ell}{2\pi}\int_0^\ell d\sigma\, T_{--}\, e^{2\pi i k(\tau+\sigma)/\ell}$$

In terms of oscillator modes,

$$L_k = \frac{T\ell}{2\pi}\left(\frac{\pi}{\ell^2 T}\right)\sum_{m,n}\alpha_m\cdot\alpha_n\, e^{-2\pi i(n+m-k)\tau/\ell}\underbrace{\int_0^\ell d\sigma\, e^{2\pi i(n+m-k)\sigma/\ell}}_{\ell\,\delta_{n+m-k,0}}$$

Therefore,

$$\boxed{L_k = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{k-n}\cdot\alpha_n\,,\qquad \bar{L}_k = \frac{1}{2}\sum_{n=-\infty}^{\infty}\bar{\alpha}_{k-n}\cdot\bar{\alpha}_n}$$

In terms of the L_k & $\bar{L}_k$,

$$T_{++} = \frac{2\pi}{\ell^2 T}\sum_k \bar{L}_k\, e^{-2\pi i k(\tau+\sigma)/\ell},$$

$$T_{--} = \frac{2\pi}{\ell^2 T}\sum_k L_k\, e^{-2\pi i k(\tau-\sigma)/\ell}$$

To insure that X^μ describe physical motions of string, we have to impose the constraints $T_{++}=T_{--}=0$, or

$$L_k = 0\,,\qquad \bar{L}_k = 0\qquad \forall k$$

For $k=0$, $\quad L_0 = 0, \quad \bar{L}_0 = 0$

or

$$L_0 + \bar{L}_0 = 0, \qquad L_0 - \bar{L}_0 = 0$$

But, $$L_0 = \frac{1}{2}\left(\frac{p^\mu p_\mu}{4\pi T} + \sum_{n\neq 0} \alpha_{-n}\cdot\alpha_n\right)$$

$$\bar{L}_0 = \frac{1}{2}\left(\frac{p^\mu p_\mu}{4\pi T} + \sum_{n\neq 0} \bar{\alpha}_{-n}\cdot\bar{\alpha}_n\right)$$

since $\alpha_0^\mu = \bar{\alpha}_0^\mu = p^\mu/\sqrt{4\pi T}$. Using $p^\mu p_\mu = -M^2$, we get:

$$\boxed{\begin{aligned} L_0 - \bar{L}_0 = 0 \quad &\Rightarrow \quad \sum_{n\neq 0} \alpha_{-n}\cdot\alpha_n = \sum_{n\neq 0} \bar{\alpha}_{-n}\cdot\bar{\alpha}_n \\ L_0 + \bar{L}_0 = 0 \quad &\Rightarrow \quad M^2 = (2\pi T)\sum_{n\neq 0}\left(\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n\right) \end{aligned}}$$

Thus these constraints (i) relate the left and right moving oscillations of the closed string and (ii) associate a rest mass M to each oscillation state of the closed string, with the units of mass fixed in terms of the string tension T. The only massless state is the non-oscillating string which collapses to zero size under the effect of the constant tension T. If the string has a fixed mass per unit length, this state becomes massless. We will find out below that in the quantum theory it is possible to get oscillating strings of zero mass.

What is the significance of the remaining constraints $L_k = 0$, $\bar{L}_k = 0$ for $k \neq 0$? Due to the Lorentz covariant nature of the formalism, the solution X^μ also contains unphysical degrees of freedom, for example, back and forth oscillations in the time direction. The remaining constraints insure that such unphysical oscillations do not affect the physics.

Note that in the case of closed strings, the solution X^μ is naturally expressed in terms of functions $e^{2\pi i n\sigma/\ell}$ that form a complete orthogonal basis over the length of the closed string $0 \leq \sigma \leq \ell$. This is no longer the case with open strings.

Oscillator Expansion of the Open String solution with Neumann Boundary Conditions

Below we apply the open string Neumann boundary conditions

$$\partial_\sigma X^\mu\Big|_{\sigma=0} = 0, \qquad \partial_\sigma X^\mu\Big|_{\sigma=\bar{l}} = 0$$

to the general solution,

$$X^\mu = x^\mu + \frac{1}{\bar{l}T} p^\mu \tau + \frac{1}{\bar{l}T} w^\mu \sigma + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \left(\frac{\alpha_n^\mu}{n} e^{-2\pi i n(\tau-\sigma)/l} + \frac{\bar{\alpha}_n^\mu}{n} e^{-2\pi i n(\tau+\sigma)/l} \right)$$

$\underline{\partial_\sigma X^\mu\Big|_{\sigma=0} = 0} \Rightarrow$

$$\frac{1}{\bar{l}T} w^\mu + \frac{\sqrt{\pi}}{l\sqrt{T}} \sum_{n\neq 0} \left(\bar{\alpha}_n^\mu - \alpha_n^\mu\right) e^{-2\pi i n\tau/l} = 0 \qquad (\forall \tau)$$

$\Rightarrow$

$$w^\mu = 0, \qquad \bar{\alpha}_n^\mu = \alpha_n^\mu .$$

Then, $\partial_\sigma X^\mu\Big|_{\sigma=\bar{l}} = 0 \Rightarrow \boxed{\bar{l} = l/2}$

Thus, the boundary condition at $\sigma = \bar{l}$, fixes the parametric length of the open string ($\bar{l}$) in terms of the parameter l that appears in the solution. We see that the solution is naturally expressed in terms of functions $e^{2\pi i n\sigma/l}$ that form a complete orthogonal basis over an interval <u>twice the length of the open</u>

string, $l = 2\bar{l}$. This means that before using the orthogonality and completeness properties of $e^{2\pi i n\sigma/l}$, we have to make sure that the range of σ is suitably extended from $[0, \bar{l} = l/2]$ to $[0, l]$ or equivalently, to $[-l/2, l/2]$. As we will see, this is always possible for computing quantities of our interest.

The open string solution with Neumann b.c. becomes:

$$\boxed{\begin{aligned} X^\mu &= x^\mu + \frac{2}{lT} p^\mu \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n}\left(e^{2\pi i n\sigma/l} + e^{-2\pi i n\sigma/l}\right) e^{-2\pi i n\tau/l} \\ &= x^\mu + \frac{2}{lT} p^\mu \tau + \frac{i}{\sqrt{\pi T}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} \cos(2\pi n\sigma/l)\, e^{-2\pi i n\tau/l} \end{aligned}}$$

$$\left(0 \leq \sigma \leq l/2 \right)$$

"Center of mass" momentum: $\mathbf{p}^\mu = T\int_0^{l/2} \partial_\tau X^\mu d\sigma = p^\mu$.

Note that $\boxed{X^\mu(\sigma,\tau) = X^\mu(-\sigma,\tau)}$ (very usefull!)

Also,

$$\partial_+ X^\mu = \frac{\sqrt{\pi}}{l\sqrt{T}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu\, e^{-2\pi i n(\tau+\sigma)/l}, \quad \left(\alpha_0^\mu = \frac{p^\mu}{\sqrt{\pi T}}\right)$$

$$\partial_- X^\mu = \frac{\pi}{l\sqrt{T}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu\, e^{-2\pi i n(\tau-\sigma)/l}$$

$\Rightarrow$

$$T_{++} = \frac{\pi}{l^2 T} \sum_{n,m} \alpha_n \cdot \alpha_m\, e^{-2\pi i (n+m)(\tau+\sigma)/l}$$

$$T_{--} = \frac{\pi}{l^2 T} \sum_{n,m} \alpha_n \cdot \alpha_m\, e^{-2\pi i (n+m)(\tau-\sigma)/l}$$

One can now compute the conserved charges associated with conformal transformations $\delta\sigma^{\pm} = \varepsilon^{\pm}(\sigma^{\pm})$,

$$Q \sim \int_0^{\bar{\ell}=\ell/2} d\sigma \left(T_{++}\varepsilon^{+} + T_{--}\varepsilon^{-}\right)$$

This is conserved $\left(\frac{\partial Q}{\partial\tau}=0\right)$ provided, $\left(T_{++}\varepsilon^{+} - T_{--}\varepsilon^{-}\right)\Big|_{\substack{\sigma=0\\ \sigma=\bar{\ell}}} = 0$

Using the N boundary conditions $\partial_\sigma X^\mu\Big|_{\substack{\sigma=0\\ \sigma=\bar{\ell}}} = 0$, this is

$$\partial_\tau X\cdot\partial_\tau X\left(\varepsilon^{+}-\varepsilon^{-}\right)\Big|_{\substack{\sigma=0\\ \sigma=\bar{\ell}}} = 0 \quad \text{or} \quad \left(\varepsilon^{+}-\varepsilon^{-}\right)\Big|_{\substack{\sigma=0\\ \sigma=\bar{\ell}=\ell/2}} = 0, \ \forall\tau .$$

$\varepsilon^{\pm}$ satisfying these boundary conditions are given by,

$$\boxed{\varepsilon^{+}(\sigma^{+}) = \sum_k \varepsilon_k e^{2\pi i k(\tau+\sigma)/\ell}, \quad \varepsilon^{-}(\sigma^{-}) = \sum_k \varepsilon_k e^{2\pi i k(\tau-\sigma)/\ell}}$$

The conserved quantity corresponding to transformation generated by the mode ε_k is

$$L_k = \frac{T\ell}{2\pi}\int_0^{\bar{\ell}} d\sigma \left(T_{++}(\sigma,\tau)\, e^{2\pi i k(\tau+\sigma)/\ell} + T_{--}(\sigma,\tau)\, e^{2\pi i k(\tau-\sigma)/\ell}\right)$$

$$= \frac{T\ell}{2\pi}\left[\int_0^{\bar{\ell}} d\sigma\, T_{++}(\sigma,\tau)\, e^{2\pi i k(\tau+\sigma)/\ell} + \int_{-\bar{\ell}}^{0} d\sigma\, T_{--}(-\sigma,\tau)\, e^{2\pi i k(\tau+\sigma)/\ell}\right]$$

But $X^\mu(\sigma) = X^\mu(-\sigma)$ & $\partial_- \xrightarrow{\sigma\to-\sigma} \partial_+ \Rightarrow T_{--}(-\sigma,\tau) = T_{++}(\sigma,\tau)$

Hence,

$$L_k = \frac{T\ell}{2\pi}\int_{-\bar{\ell}}^{\bar{\ell}} d\sigma\, T_{++}\, e^{2\pi i k(\tau+\sigma)/\ell}$$

Now that the range of σ has effectively been extended to $-\frac{l}{2} \leq \sigma \leq \frac{l}{2}$, we can use the orthogonality property of $e^{2\pi i n\sigma/l}$ to evaluate the integral.

$$L_k = \frac{\cancel{T} l}{2\cancel{\pi}} \frac{\cancel{\pi}}{l^2 \cancel{T}} \sum_{n,m} \alpha_n \cdot \alpha_m e^{-2\pi i(n+m-k)\tau/l} \underbrace{\int_{-l/2}^{l/2} d\sigma\, e^{-2\pi i(n+m-k)\sigma/l}}_{l\,\delta_{n+m-k,0}}$$

$$\boxed{L_k = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{k-n} \cdot \alpha_n}$$

The constraints $T_{\alpha\beta} = 0$ become $L_k = 0 \;\forall k$.

For $k=0$,

$$L_0 = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \frac{1}{2}\sum_{n\neq 0} \alpha_{-n} \cdot \alpha_n = \frac{1}{2\pi T} p^\mu p_\mu + \frac{1}{2}\sum_{n\neq 0} \alpha_{-n} \cdot \alpha_n$$

$$\boxed{L_0 = 0 \Rightarrow \quad M^2 = (\pi T) \sum_{n\neq 0} \alpha_{-n} \cdot \alpha_n}$$

Again, at the classical level, all oscillating modes of the string acquire a non-zero mass. This will change in the quantum theory.