Classical Symmetries of Polyakov Action

We describe three main symmetry groups of the Polyakov action,

$$S_p = -\frac{T}{2}\int d^2\sigma \sqrt{-\det(h)}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X_\mu .$$

(a) Global Symmetries:

The flat D-dimensional space-time is invariant under translations and Lorentz transformations, which together form the Poincaré group:

i) $X^\mu \longrightarrow X^\mu + b^\mu$ (translations)

ii) $X^\mu \longrightarrow X^\mu + \omega^\mu{}_\nu X^\nu$, $(\omega_{\mu\nu} = -\omega_{\nu\mu})$.
(infinitesimal Lorentz rotations)

(i) Leads to the conservation of energy and momentum, while (ii) leads to the conservation of angular momentum in D-dimensions. These transformations keep Sp invariant. The fact that (i) and (ii) do not depend on σ and τ means that in string theory the Poincaré group in D-dimensions appears as a global internal symmetry group that acts on the fields X^μ but not on the 2-dimensional space-time (σ, τ).

(b) 2-dim. General Coordinate Transformations (GCT)

These are also known as reparametrizations or diffeomorphisms in 2-dimensions:

$$\boxed{\sigma^\alpha \longrightarrow \tilde{\sigma}^\alpha(\sigma,\tau)}$$

Then:

$$\frac{\partial}{\partial\tilde{\sigma}^\alpha} = \frac{\partial\sigma^\rho}{\partial\tilde{\sigma}^\alpha}\frac{\partial}{\partial\sigma^\rho} \qquad \left(\text{or } \tilde{\partial}_\alpha = \frac{\partial\sigma^\rho}{\partial\tilde{\sigma}^\alpha}\partial_\rho\right)$$

$$\tilde{h}^{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\tilde{\sigma}^\alpha}{\partial\sigma^\rho}\frac{\partial\tilde{\sigma}^\beta}{\partial\sigma^\lambda} h^{\rho\lambda}(\sigma), \quad \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma)$$

$$\Rightarrow \boxed{\tilde{h}^{\alpha\beta}\tilde{\partial}_\alpha\tilde{X}^\mu \tilde{\partial}_\beta\tilde{X}_\mu = h^{\rho\lambda}\partial_\rho X^\mu \partial_\lambda X_\mu}$$

Similarly,

$$\det\left(\frac{\partial\tilde{\sigma}^\alpha}{\partial\sigma^\rho}\right) = J \quad (\text{Jacobian})$$

and, $d^2\tilde{\sigma} = J\, d^2\sigma$

$$\det(\tilde{h}_{\alpha\beta}) = J^{-2}\det(h_{\alpha\beta})$$

$$\Rightarrow \boxed{d^2\tilde{\sigma}\sqrt{-\det(\tilde{h})} = d^2\sigma\sqrt{-\det(h)}}$$

This shows that S_p is invariant.

Infinitesimal form:

$$\boxed{\sigma^\alpha \Longrightarrow \tilde{\sigma}^\alpha = \sigma^\alpha + \varepsilon^\alpha(\sigma)} \quad (2 \text{ parameters: } \varepsilon^0, \varepsilon^1)$$

(c) Weyl invariance:

$$h_{\alpha\beta} \longrightarrow e^{\rho(\sigma)} h_{\alpha\beta} \quad \text{(Weyl Scaling)}$$

under this transformation,

$$\left.\begin{aligned} h^{\alpha\beta} &\longrightarrow e^{-\rho} h^{\alpha\beta} \\ \det(h) &\rightarrow e^{+2\rho} \det(h) \end{aligned}\right\} \sqrt{-\det(h)}\, h^{\alpha\beta} \text{ is invariant.}$$

Physical meaning:

$$ds^2_{ws} = h_{\alpha\beta}\, d\sigma^\alpha d\sigma^\beta \rightarrow e^{\rho} ds^2_{ws}$$

This transformation has 1 parameter $\rho(\sigma)$

Consequences of Classical Symmetries

Consequence of Weyl invariance:

Let us parametrize the worldsheet metric as

$$h^{\alpha\beta} = e^{-\phi} \hat{h}^{\alpha\beta}$$

As a consequence of Weyl invariance, S_P does not depend on $\phi(\sigma)$, $\dfrac{\delta S_P}{\delta \phi} = 0$

But,

$$\frac{\delta S_P}{\delta \phi} = \frac{\delta S}{\delta h^{\alpha\beta}} \frac{\delta h^{\alpha\beta}}{\delta \phi} = (\#)\sqrt{-\det(h)}\, T_{\alpha\beta}\left(-h^{\alpha\beta}\right)$$

$$= (\#)\, h^{\alpha\beta} T_{\alpha\beta} = T^{\alpha}{}_{\alpha} = 0$$

Hence, the <u>tracelessness</u> of $T_{\alpha\beta}$ is a consequence of Weyl invariance.
$T_{\alpha\beta}$ has only 2 independent components and the equations $T_{\alpha\beta} = 0$ give us only 2 constraints.

<u>Consequence of Reparametrization invariance:</u>

Consider an infinitesimal reparametrization:

$$\sigma^\alpha \to \tilde{\sigma}^\alpha = \sigma^\alpha + \varepsilon^\alpha(\sigma)$$

We show that the invariance of S_P implies:

$$\boxed{\nabla^\alpha T_{\alpha\beta} = 0}$$

For this, we have to calculate the variation of $h_{\alpha\beta}$,

$$\bar{\delta} h_{\alpha\beta} = \tilde{h}_{\alpha\beta}(\sigma) - h_{\alpha\beta}(\sigma)$$

$$\tilde{h}^{\alpha\beta}(\tilde{\sigma}) \equiv \tilde{h}^{\alpha\beta}(\sigma+\varepsilon) = \frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^\lambda}\frac{\partial \tilde{\sigma}^\beta}{\partial \sigma^\rho} h^{\lambda\rho}(\sigma) = (\delta^\alpha_\lambda + \partial_\lambda \varepsilon^\alpha)(\delta^\beta_\rho + \partial_\rho \varepsilon^\beta) h^{\lambda\rho}(\sigma)$$

or to first order in ε^α,

$$\tilde{h}^{\alpha\beta}(\tilde{\sigma}) + \partial_\lambda h^{\alpha\beta}\varepsilon^\lambda = h^{\alpha\beta}(\sigma) + \partial^\alpha\varepsilon^\beta + \partial^\beta\varepsilon^\alpha$$

$\Rightarrow$

$$\tilde{h}^{\alpha\beta}(\sigma) - h^{\alpha\beta}(\sigma) = \partial^\alpha\varepsilon^\beta + \partial^\beta\varepsilon^\alpha - \partial_\lambda h^{\alpha\beta}\varepsilon^\lambda = \nabla^\alpha\varepsilon^\beta + \nabla^\beta\varepsilon^\alpha$$

$$\therefore \quad \boxed{\bar{\delta} h^{\alpha\beta} = \nabla^\alpha\varepsilon^\beta + \nabla^\beta\varepsilon^\alpha}$$

$$\left(\nabla^\alpha\varepsilon^\beta = h^{\alpha\alpha'}\nabla_{\alpha'}\varepsilon^\beta \,, \quad \nabla_{\alpha}\varepsilon^\beta = \partial_\alpha\varepsilon^\beta + \Gamma^\beta_{\alpha\gamma}\varepsilon^\gamma \right)$$

We can choose $\varepsilon^\alpha(\sigma)$ to be a local variation so that it vanishes at the boundary of the worldsheet. Then, since σ^α are dummy integration variables,

$$S[\tilde{h}(\tilde{\sigma}), \tilde{X}(\tilde{\sigma})] = S[\tilde{h}(\sigma), \tilde{X}(\sigma)]$$

so that

$$\delta S = S[\tilde{h}(\sigma), \tilde{X}(\sigma)] - S[h(\sigma), X(\sigma)]$$

$$= S[h + \bar{\delta}h, X + \bar{\delta}X] - S[h, X]$$

$$= \int \left(\frac{\delta S}{\delta h^{\alpha\beta}} \bar{\delta}h^{\alpha\beta} + \frac{\delta S}{\delta X^\mu} \bar{\delta}X^\mu \right) d^2\sigma$$

$$= (\ast) \int d^2\sigma \sqrt{-\det(h)}\; T_{\alpha\beta}\left(\nabla^\alpha\varepsilon^\beta + \nabla^\beta\varepsilon^\alpha\right)$$

$$\boxed{\delta S = 2(\ast) \int d^2\sigma \sqrt{-\det(h)}\; T_{\alpha\beta}\nabla^\alpha\varepsilon^\beta}$$

where the eqn. of motion for X^μ is assumed to be satisfied, $\frac{\delta S}{\delta X^\mu} = 0$

Now, use

$$\nabla^\alpha (T_{\alpha\beta}\varepsilon^\beta) = (\nabla^\alpha T_{\alpha\beta})\varepsilon^\beta + T_{\alpha\beta}(\nabla^\alpha \varepsilon^\beta)$$

and $\sqrt{-\det(h)}\,\nabla^\alpha B_\alpha = \partial_\alpha\left(\sqrt{-\det(h)}\,B^\alpha\right)$

to get:

$$\delta S = (\gamma) \int d^2\sigma \sqrt{-\det(h)} \left[\nabla^\alpha (T_{\alpha\beta}\varepsilon^\beta) - (\nabla^\alpha T_{\alpha\beta})\varepsilon^\beta\right]$$

$$= (\gamma) \underbrace{\int d^2\sigma\, \partial^\alpha\left(\sqrt{-\det(h)}\, T_{\alpha\beta}\varepsilon^\beta\right)}_{=0 \text{ (boundary term)}} - \int d^2\sigma \sqrt{-\det(h)} (\nabla^\alpha T_{\alpha\beta})\varepsilon^\beta$$

$\therefore$

$$\delta S = 0 \Rightarrow \boxed{\nabla^\alpha T_{\alpha\beta} = 0}$$

(Note(1): The boundary term in the above expression vanishes since we take $\varepsilon^\beta(\sigma)$ to be zero on the boundary)

Note(2): $\nabla^\alpha T_{\alpha\beta} = 0$ does not imply energy and momentum conservation. This is consistent with the fact that when $h_{\alpha\beta}(\sigma) \neq \eta_{\alpha\beta}$, then the metric is not invariant under translations in σ and τ and hence one does not expect to obtain a corresponding conservation law. A conservation law follows from $\partial^\alpha T_{\alpha\beta} = 0$, which is not the case with a general $h_{\alpha\beta}(\sigma)$.

Gauge Fixing the Polyakov Action

The Polyakov action contains $h_{\alpha\beta}$ with its three redundant degrees of freedom. These can be removed (or "gauged away") by using the 2 parameters of the GCT and the 1 parameter of Weyl scaling.

Starting from any $h_{\alpha\beta}$, it is always possible to find a general coordinate transformation that brings it to the form

$$(h_{\alpha\beta}) \xrightarrow{\text{GCT}} e^{\phi(\sigma)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \equiv e^{\phi(\sigma)} (\eta_{\alpha\beta})$$

which removes 2-degrees of freedom from $h_{\alpha\beta}$. (Here we are only dealing with worldsheets of simple topology. The general case is more involved, but very similar.)

Now, we can use Weyl scaling to eliminate the scale factor,

$$(h_{\alpha\beta}) \xrightarrow{\text{GCT + Weyl}} (\eta_{\alpha\beta})$$

Since both these transformations are symmetries of the action, we have:

$$\boxed{\begin{aligned} S_P &= \frac{-T}{2}\int d^2\sigma \; \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \\ &= \frac{T}{2}\int d^2\sigma \left(\partial_\tau X^\mu \partial_\tau X_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu\right) \end{aligned}}$$

Thus we get a theory of D massless, free scalar fields in 2 dimensions, which is one of the simplest field theories one could write down.

This simple field theory has to be supplemented by the condition

$$\boxed{T_{\alpha\beta} = 0}$$

Now that since $h_{\alpha\beta}$ has been fixed to the constant matrix $\eta_{\alpha\beta}$, the above condition is interpreted as a constraint in the 2-d flat space-time (rather than an equation of motion). Furthermore, $\nabla^\alpha T_{\alpha\beta} = 0$ reduces to

$$\boxed{\partial^\alpha T_{\alpha\beta} = 0}$$

In flat space, $T_{\alpha\beta}$ can be obtained (along with its conservation law) from the translation and Lorentz invariances of the S_P (with $h^{\alpha\beta} = \eta^{\alpha\beta}$) through Noether's theorem. (This will be explained later)

Explicitly, $T_{\alpha\beta}$ in flat space becomes:

$$\boxed{T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\left(\eta^{\lambda\rho}\partial_\lambda X^\mu \partial_\rho X_\mu\right)}$$

Using the notation

$$\frac{\partial X^\mu}{\partial \tau} = \dot{X}^\mu, \qquad \frac{\partial X^\mu}{\partial \sigma} = X'^\mu,$$

one has

$$T_{\alpha\beta} = \begin{pmatrix} \frac{1}{2}\left(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right) & \dot{X}^\mu X'_\mu \\ X'^\mu \dot{X}_\mu & \frac{1}{2}\left(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right) \end{pmatrix}$$

Instead of the coordinates (σ, τ) it is extremely useful to use a new set of coordinates (σ^+, σ^-):

$$\sigma^+ = \tau + \sigma, \qquad \sigma^- = \tau - \sigma$$

which implies

$$\frac{\partial}{\partial\sigma} = \frac{\partial}{\partial\sigma^+} - \frac{\partial}{\partial\sigma^-}, \qquad \frac{\partial}{\partial\tau} = \frac{\partial}{\partial\sigma^+} + \frac{\partial}{\partial\sigma^-}$$

In the new coordinates the worldsheet metric takes the form:

$$\eta = \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}, \quad \eta^{-1} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

Raising and lowering indices using this metric gives (for example, for a 2-dimensional vector V^α):

$$\left.\begin{array}{l} V_+ = \eta_{+-} V^- = -\frac{1}{2} V^- \\ V_- = \eta_{-+} V^+ = -\frac{1}{2} V^+ \end{array}\right\} \quad \begin{array}{l} V^- = -2 V_+ \\ V^+ = -2 V_- \end{array}$$

Now we can rewrite all expressions in terms of $\sigma^\pm$. For this, one can either use the standard formulae for coordinate transformations, or simply note that our expressions with indices (α, β) are tensor expressions and therefore hold in any coordinate system (provided we use the appropriate form of the metric $\eta_{\alpha\beta}$). Therefore in these expressions we can simply interpret (α, β) as taking values $\pm$ (rather than 0,1).
Thus one obtains:

$$\boxed{\begin{array}{c} S_P = 2T \int d^2\sigma \; \partial_+ X^\mu \partial_- X_\mu \\ T_{++} = \partial_+ X^\mu \partial_+ X_\mu \, , \qquad T_{--} = \partial_- X^\mu \partial_- X_\mu \\ T_{+-} = T_{-+} = 0 \end{array}}$$

Also, $\partial^\alpha T_{\alpha\beta} = 0 \Rightarrow$

$$\partial^+ T_{++} = 0 \quad \Rightarrow \quad \boxed{\partial_- T_{++} = 0}$$

$$\partial^- T_{--} = 0 \quad \Rightarrow \quad \boxed{\partial_+ T_{--} = 0}$$

Note that $T_{+-} = T_{-+} = 0$ is the statement of tracelessness of $T_{\alpha\beta}$:

$$Tr(T) = T^{\alpha}{}_{\alpha} = T^{+}{}_{+} + T^{-}{}_{-} = -2\left(T_{-+} + T_{+-}\right) = 0$$

The condition $\partial^{\alpha} T_{\alpha\beta} = 0$ then implies that T_{++} (T_{--}) is a function of σ^{+} (σ^{-}) alone:

$$\boxed{T_{++} = T_{++}(\sigma^{+}), \qquad T_{--} = T_{--}(\sigma^{-})}$$

The constraint $T_{\alpha\beta} = 0$ boils down to two non-trivial equations

$$\boxed{T_{++} = 0 \;, \quad T_{--} = 0}$$

These constraints are called the "Virasoro" constraints.

Conformal Invariance

The 3 unphysical degrees of freedom in $h_{\alpha\beta}$ have been eliminated by using the freedom associated with 2 reparametrization and 1 scale transformations. In the process, $h_{\alpha\beta}$ has been fixed to

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \quad \text{in} \quad (\sigma, \tau) \text{ coordinates}$$

or, equivalently, as

$$\eta = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix} \quad \text{in} \quad \sigma^+, \sigma^- \text{ coordinates.}$$

Generic scalings and reparametrizations will change the metric back to some $h_{\alpha\beta}$ and hence are not symmetries of the gauge fixed action. In this sense, these transformations have been fixed. There are, however, specific combinations of Weyl and general coordinate transformations that keep η invariant. Consider an infinitesimal reparametrization $\tilde{\sigma}^\alpha = \sigma^\alpha + \varepsilon^\alpha$ such that

$$\eta_{\alpha\beta} \longrightarrow \tilde{h}_{\alpha\beta} = e^{-\Lambda} \eta_{\alpha\beta} \qquad (\text{for some } \Lambda(\sigma)).$$

The scale factor $e^{-\Lambda}$ can now be removed by an appropriate Weyl scaling:

$$e^{-\Lambda}\eta_{\alpha\beta} \longrightarrow e^{\Lambda}(e^{-\Lambda}\eta_{\alpha\beta}) = \eta_{\alpha\beta}$$

This combination of transformations is called a conformal transformation. Explicitly,

$$\tilde{h}^{\tilde{\alpha}\tilde{\beta}} = \frac{\partial\tilde{\sigma}^{\tilde{\alpha}}}{\partial\sigma^{\alpha}}\frac{\partial\tilde{\sigma}^{\tilde{\beta}}}{\partial\sigma^{\beta}}\eta^{\alpha\beta} \simeq \eta^{\tilde{\alpha}\tilde{\beta}} + \partial^{\tilde{\beta}}\varepsilon^{\tilde{\alpha}} + \partial^{\tilde{\alpha}}\varepsilon^{\tilde{\beta}} + \cdots$$

$$\simeq \eta^{\tilde{\alpha}\tilde{\beta}} + \Lambda\eta^{\tilde{\alpha}\tilde{\beta}} + \cdots$$

or, to first order,

$$\boxed{\partial^{\alpha}\varepsilon^{\beta} + \partial^{\beta}\varepsilon^{\alpha} = \Lambda\eta^{\alpha\beta}}$$

It is convenient to work in the σ^+, σ^- system where $\eta^{++} = 0$, $\eta^{--} = 0 \Rightarrow \partial^+\varepsilon^+ = 0$ and $\partial^-\varepsilon^- = 0$, or

$$\boxed{\partial_-\varepsilon^+ = 0 \quad , \quad \partial_+\varepsilon^- = 0}$$

Thus, any $\varepsilon^+(\sigma^+)$ and $\varepsilon^-(\sigma^-)$ lead to conformal transformations.

From this it follows that finite conformal transformations are given by:

$$\boxed{\begin{gathered}\sigma^+ \longrightarrow \tilde{\sigma}^+ = f(\sigma^+), \quad \sigma^- \longrightarrow \tilde{\sigma}^- = g(\sigma^-) \\ \eta_{\alpha\beta} \longrightarrow \eta_{\alpha\beta}\end{gathered}}$$

with f and g being arbitrary functions of σ^+ & σ^-.

The Weyl scaling has already been used to remove the scale factor from the metric. Thus effectively, a conformal transformation amounts to a "holomorphic" transformation of σ^+ and σ^-, keeping $\eta_{\alpha\beta}$ unchanged. Since all these transformations are symmetries of the gauge fixed Polyakov action, conformal transformations keep S_P invariant.

Conformal invariance is of great importance in string theory: it implies that objects in the theory are classified according to their behaviour under these transformations, much in the same way that rotational invariance in quantum mechanics leads to the classification of states based on their angular momentum quantum numbers. Because of this, we will spend a little more time on the subject before moving on to other aspects of the theory.