Introduction to Bosonic String Theory

Though we are mainly interested in Superstring theory, it is best to start the discussion with bosonic strings. This will allow us to introduce some of the basic concepts of string theory without getting entangled in the complications of superstrings.

First we start with the classical theory: the string action and its symmetries, gauge fixing the symmetries, conformal invariance, constraints, equations of motion and the importance of boundary conditions. Only after mastering those concepts, we will move on to the quantum theory.

The classical Theory of Bosonic Strings

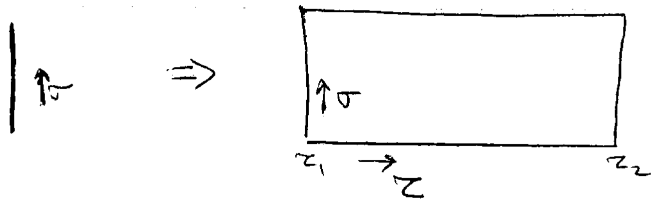
Basic concepts:

A string is a 1-dimensional object moving in D-dimensional space-time (1 time, D-1 space dimensions). As the string moves in time, it sweeps a 2-dimensional are in the D-dimensional space-time. This is called the worldsheet of the string (in analogy with the world line of a particle).

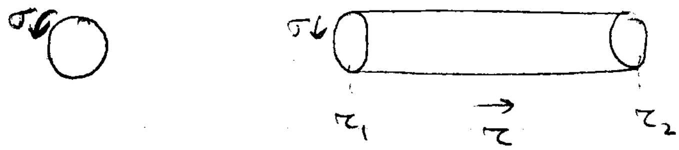
The string is parametrized by a parameter σ, $0 \leq \sigma \leq l$ (for some finite l).

As the string moves in time, every point on the string describes a trajectory in space-time. This trajectory can be parametrized by a variable τ. τ could have the range $-\infty < \tau < +\infty$. Hence, every point on the worldsheet is parametrized by the pair (σ, τ). Strings could be <u>open</u> or <u>closed</u>

open string:



closed string:



Notation: σ^α, $\{\alpha = 0, 1\}$,

$$\left(\begin{array}{c} \sigma^0 = \tau \\ \sigma^1 = \sigma \end{array} \right)$$

Let X^μ $(\mu = 0, 1, \ldots D-1)$ be the coordinates in a flat D-dimensional space-time.

The space-time position of a point (σ, τ) on the string worldsheet is given by the functions <u>$X^\mu(\sigma, \tau)$</u>. We say that these functions <u>embed</u> the worldsheet in space time.

Note: σ & τ are mathematical parameters and do not have much physical significance. However, the physical shape of the world sheet is described by the functions:

$$X^\mu(\sigma,\tau), \quad (\mu = 0, 1, \cdots D-1)$$

These tell us how the string moves and hence are the dynamical variables. Our intention is to find the equations that govern the behaviour of these functions.

Different notions of "distance"

We now describe the different notions of distance that can be defined and explain the concept of "induced metric".

(a) The "Lorentzian" distance between any two adjacent points in space-time is given by:

$$ds^2_{ST} = \eta_{\mu\nu}\, dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^{D-1})^2$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots \\ 0 & +1 & 0 & \cdots & \vdots \\ 0 & 0 & +1 & \cdots & \\ \cdots & & & & +1 \end{pmatrix}$$: flat space time metric

(b) One the worldsheet, we can define an "intrinsic" Lorentzian distance:

$$ds_{ws}^2 = -d\tau^2 + d\sigma^2 \equiv \eta_{\alpha\beta}\, d\sigma^\alpha d\sigma^\beta$$

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Here, we have chosen a flat metric for the (σ,τ) space. This notion of distance is intrinsic in the sense that it does not **depend** on the space-time embedding of the worldsheet.

Since σ^α are mathematical parameters, this notion of distance does not have a physical significance.

(c) One can also measure the space-time distance between two adjacent points on the worldsheet:

$$ds^2 = \eta_{\mu\nu}\, dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}\, d\sigma^\alpha d\sigma^\beta$$

$$= g^{(in)}_{\alpha\beta}\, d\sigma^\alpha d\sigma^\beta$$

This is as if we had a metric on the worldsheet:

$$\boxed{g^{(in)}_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} = \begin{pmatrix} \dot{X}^\mu \dot{X}_\mu & \dot{X}^\mu X'_\mu \\ \dot{X}^\mu X'_\mu & X'^\mu X'^\mu \end{pmatrix}}$$

$g^{(in)}_{\alpha\beta}$ is called the "induced" metric (the metric induced on the worldsheet by the embedding).

This is the physically relevant notion of distance on the world sheet.

The Nambu-Goto Action

Dynamical variables: $X^\mu(\sigma, \tau)$

In physics the equations of motion for the dynamical variables of the theory are derived from an action (S) by using the least action principle.

So, we have to construct an action for the $X^\mu(\sigma, \tau)$. The most obvious choice is to take the area of the worldsheet as the action (in analogy with the length of the worldline for a particle). This will give us the Nambu-Goto action for the string:

$$\boxed{S_{NG} = T \int d^2\sigma \sqrt{-\det(g^{(in)})}}$$

$[T] = M^2$

$$\det(g^{(in)}) = (\dot{X}^\mu \dot{X}_\mu)(X'^\mu X'_\mu) - (\dot{X}^\mu X'_\mu)^2$$

$$= (\dot{X})^2 (X')^2 - (\dot{X} \cdot X')^2$$

This describes a 2-dim. field theory of D fields X^μ. However the action is not polynomial and hence quantization is more complicated.

special case of S_{NG} : point particle:

$$ds^2 = \eta_{\mu\nu}\, \partial_\tau X^\mu \partial_\tau X^\nu d\tau^2 = g_{\tau\tau}\, d\tau^2 .$$

$$\int ds = \int d\tau \sqrt{-g_{\tau\tau}} = \int \sqrt{-\eta_{\mu\nu}\, \partial_\tau X^\mu \partial_\tau X^\nu}$$

$$= c \int \sqrt{1 - \frac{\vec{v}^2}{c^2}}\, dt \qquad \tau = X^0 = ct$$

$$v^i = \partial_t X^i$$

The Polyakov Action :

Let us write the simplest theory of D scalar fields X^μ in a 2-dimensional space (σ, τ). in general the two dimensional space can have some metric $h_{\alpha\beta}(\sigma,\tau)$, so that now the intrinsic distance on the worldsheet is

$$ds^2_{ws} = h_{\alpha\beta}\, d\sigma^\alpha d\sigma^\beta$$

The simplest action is

$$\boxed{S_P = -\frac{T}{2}\int d^2\sigma \sqrt{-\det(h)}\; h^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}}$$

(The polyakov action)

Note: we have introduced 3 extra degrees of freedom in $h_{\alpha\beta}(\sigma,\tau)$. These should not have a physical significance. Indeed they are removed by using the equations of motion for $h_{\alpha\beta}$:

$$\frac{\delta S}{\delta h^{\alpha\beta}} = 0$$

Once this equation is solved, S_P will reduce to S_{NG} !

Digression: The stress-energy tensor is defined as

$$\boxed{T_{\alpha\beta} = \frac{-2}{T}\frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}}}$$

$$\left(\left(\sqrt{-h} \equiv \sqrt{-\det(h)}\right)\right)$$

To evaluate this, we need $\dfrac{\partial\sqrt{-h}}{\partial h^{\alpha\beta}}$:

For a matrix h,

$$\det(h) = e^{\mathrm{tr}\ln(h)}$$

$$\det(h+\delta h) = e^{\mathrm{tr}\ln(h+\delta h)} = e^{\mathrm{tr}\ln[h(\mathbb{1}+h^{-1}\delta h)]}$$

$$= e^{\mathrm{tr}\ln h + \mathrm{tr}\ln(\mathbb{1}+h^{-1}\delta h)}$$

$\boxed{\ln(1+x)\sim x+\cdots}$

$$= \det(h)\, e^{\mathrm{tr}\ln(\mathbb{1}+h^{-1}\delta h)}$$

$$\approx \det(h)\, e^{\mathrm{tr}(h^{-1}\delta h)} \approx \det(h)(1+\mathrm{tr}\, h^{-1}\delta h\cdots)$$

$\Rightarrow$

$$\det(h+\delta h) - \det(h) = \det(h)\, h^{\alpha\beta}(\delta h_{\alpha\beta}) + \cdots$$

$$\approx -\det(h)\, \delta h^{\alpha\beta} h_{\alpha\beta} + \cdots$$

$$\boxed{\frac{\partial \det(h)}{\partial h^{\alpha\beta}} = -\det(h)\, h_{\alpha\beta}}$$

$$\frac{\partial \sqrt{-\det(h)}}{\partial h^{\alpha\beta}} = -\frac{1}{2}\sqrt{-\det(h)}\, h_{\alpha\beta}$$

$\therefore$

$$\frac{\delta S}{\delta h^{\alpha\beta}} = \frac{-T}{2}\left(\frac{\partial\sqrt{-\det(h)}}{\partial h^{\alpha\beta}} h^{\rho\sigma} + \sqrt{-\det(h)}\frac{\partial h^{\rho\sigma}}{\partial h^{\alpha\beta}}\right)\partial_\rho X^\mu \partial_\sigma X_\mu$$

$$= \frac{-T}{2}\sqrt{-\det(h)}\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}(h^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X_\mu)\right)$$

$\therefore$

$$\boxed{T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \frac{1}{2}h_{\alpha\beta}\left(h^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\right)}$$

The trace of $T_{\alpha\beta}$ is defined as

$$Tr(T_{\alpha\beta}) = T^{\alpha}{}_{\alpha} = h^{\alpha\beta}T_{\alpha\beta}$$

since $h^{\alpha\beta}h_{\alpha\beta} = \sum_{\beta}\delta^{\beta}_{\beta} = 2$, it follows that $T_{\alpha\beta}$ is traceless,

$$\boxed{T^{\alpha}{}_{\alpha} = 0}$$

The reason for this will be explained latter.

(End of Digression)

Going back to S_P, the eqn. of motion for $h_{\alpha\beta}$ is

$$\frac{\delta S_P}{\delta h^{\alpha\beta}} = 0 \quad \Rightarrow \quad T_{\alpha\beta} = 0$$

Using the expression for $T_{\alpha\beta}$ and the definition of $g^{(in)}_{\alpha\beta}$:

$$g^{(in)}_{\alpha\beta} \equiv \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} = \frac{1}{2} h_{\alpha\beta}\left(h^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\right)$$

$$\Rightarrow \quad \det(g^{(in)}) = \frac{1}{4}\det(h)\left(h^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\right)^2$$

or

$$\sqrt{-\det(g^{(in)})} = \frac{1}{2}\sqrt{-\det(h)}\left(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\right)$$

Therefore:

$$\boxed{S_P\Big|_{T_{\alpha\beta}=0} = S_{NG}}$$

Thus after eliminating the extra 3 degrees of freedom in $h_{\alpha\beta}$ by using their equations of motion, we recover S_{NG} from S_p. This equivalence shows that instead of S_{NG}, from now on we can work with S_p which is the theory of D free massless scalar fields in 2-dimensions,