

QFT problem set 2

Due date: Monday, Jan 13, 2025

1. In Dirac theory, starting from the expression for the Noether charges, express the Hamiltonian H , momentum P_i , $U(1)$ charge Q , and spin \vec{S} operators in terms of the Dirac field and its conjugate.

Compute Q in terms of the creation and annihilation operators (*4 points*).

2. In the Dirac theory, consider a 1-particle state $|\mathbf{1}_{p,r}\rangle$ of momentum vector \vec{p} created by either $c_r^\dagger(\vec{p})$ or $d_r^\dagger(\vec{p})$. The projection of the spin operator \vec{S} along the particle momentum, $S_p = \vec{S} \cdot \hat{p}$, is given by (in units where $\hbar = 1$, and normal ordered)

$$S_p = \frac{1}{2} \int d^3x \text{N}(\psi^\dagger \sigma_p \psi)$$

(See Mandl and Shaw for notations and the identities to use in this problem). Show that in terms of the lowering and raising operators, one gets,

$$\langle \mathbf{1}_{p,r} | S_p | \mathbf{1}_{p,r} \rangle = \frac{1}{2} (-1)^{r+1} \langle \mathbf{1}_{p,r} | \left(d_r^\dagger(\vec{p}) d_r(\vec{p}) + c_r^\dagger(\vec{p}) c_r(\vec{p}) \right) | \mathbf{1}_{p,r} \rangle.$$

(Important: Note that \vec{p} is the momentum of the state, not to be confused with the dummy momentum labels, say, \vec{p}' and \vec{p}'' , that appear in the expansions of ψ and ψ^\dagger . The same holds for the index r above and the indices r' and r'' in the expansion of the Dirac fields.)

What are the spin eigenvalues when $|\mathbf{1}_{p,r}\rangle$ is either the 1-particle states $c_r^\dagger(\vec{p})|0\rangle$ or $d_r^\dagger(\vec{p})|0\rangle$ (for $r = 1, 2$)? (*4 points*)

3. Using the expansion of the Dirac fields $\psi(x)$ and $\bar{\psi}(x)$ in terms of creation and annihilation operators, compute the covariant anticommutation relations $\{\psi_\alpha^\pm(x), \bar{\psi}_\beta^\mp(y)\}$ showing that the answer is given by eqn (4.52b) of Mandl and Shaw (2nd Edition). (*3 points*)
4. Using the result of question 4, compute the fermion propagator $\langle 0 | T (\psi(x) \bar{\psi}(y)) | 0 \rangle$.

Show that the result is $iS_F(x-y)$ given by the contour integral

$$iS_F(x) = \frac{i\hbar}{(2\pi\hbar)^4} \int d^3p \int_{C_F} dp^0 e^{-ipx/\hbar} \frac{\not{p} + mc}{p^2 - m^2c^2}$$

where C_F is the Feynman contour in the complex p^0 plane (provide your reasoning and calculation details). (*3 points*)