

QFT problem set 1

Due date: Sunday March 01, 2026

(Use the metric $\eta_{00} = 1 = -\eta_{ii}$ and set $c = 1 = \hbar$)

1. The Lagrangian density for a free real scalar field $\phi(x)$ is $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$. Obtain the equation of motion for ϕ . Starting with a trial solution $\phi \sim e^{ik_\mu x^\mu}$ solve this equation and obtain the general solution (up to normalization)

$$\phi(x) = \sum_{\mathbf{k}} \left(\frac{1}{2V\omega_{\mathbf{k}}} \right)^{1/2} (a(\mathbf{k}) e^{-ikx} + a^\dagger(\mathbf{k}) e^{ikx}),$$

where $kx = k_\mu x^\mu$ and $k^0 = \omega_{\mathbf{k}} = \sqrt{m^2 + \mathbf{k}^2}$, taking $\hbar = c = 1$. (4 points)

2. Derive the following expression for the lowering operator $a(\mathbf{k})$

$$a(\mathbf{k}) = \frac{1}{(2V\omega_{\mathbf{k}})^{1/2}} \int d^3x e^{ikx} (i\dot{\phi}(x) + \omega_{\mathbf{k}}\phi(x))$$

and obtain the commutation relations for $a(\mathbf{k})$ and $a^\dagger(\mathbf{k}')$ using the equal time commutation relations for the fields (note the relation between $\dot{\phi}$ and π). (4 points)

3. a and a^\dagger are lowering and raising operators satisfying $[a, a^\dagger] = 1$, $[a, a] = 0$ and $[a^\dagger, a^\dagger] = 0$. Given a state

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle,$$

show that $\langle n|n\rangle = 1$. Furthermore, given $|n+1\rangle = C a^\dagger |n\rangle$, find C so that $|n+1\rangle$ is normalized to unity. (Hint: It is convenient, though not necessary, to first evaluate the commutator $[a, (a^\dagger)^n]$.) (4 points)

4. For the real scalar field, starting from the energy-momentum tensor, obtain the Hamiltonian H and momenta P_i in terms of $\dot{\phi}$ and $\vec{\nabla}\phi$. Using the solution for $\phi(x)$, calculate these in terms of the creation and annihilation operators, showing the steps in your calculation. (4 points).
5. For a complex scalar field $\phi(x) \neq \phi^\dagger(x)$, starting from the conserved current s^μ of global phase symmetry (equation 3.32 in Mandle & Shaw), first obtain the expression for Q in terms of fields (the expression in equation 3.31 of M&S is incorrect) and then evaluate Q in terms of the creation and annihilation operators (equation 3.34 in Mandle & Shaw). (4 points).