Abelian Gauge Theory:

$$\psi \longrightarrow e^{iq\theta}\psi, \quad \partial_\mu\theta = 0 : \text{ Global symmetry.}$$

But,

$$\theta = \theta(x) \Rightarrow \partial_\mu \longrightarrow D_\mu = \partial_\mu + iq A_\mu$$

so that

$$\mathcal{L}(\psi, \partial_\mu\psi) \text{ becomes } \mathcal{L}(\psi, D_\mu\psi).$$

This is invariant under

$$\psi \to e^{iq\theta}\psi, \qquad A_\mu \longrightarrow A_\mu - \partial_\mu\theta.$$

(Local gauge invariance). Requires introducing A_μ through the covariant derivative using the "minimal substitution" prescription.

A_μ: interaction between electric charges.

$$e^{iq\theta} \in U(1).$$

Non-Abelian Theory:

Consider:

n species of fermions: ψ_a, $a = 1, \ldots n$.

Each ψ_a is a 4-spinor: $\{\psi_{a\alpha}\}$

$$\mathcal{L}_D = \sum_{a=1}^{n}\left(i\bar{\psi}_a\gamma^\mu\partial_\mu\psi_a - m\bar{\psi}_a\psi_a\right)$$

Note: All ψ_a are assigned the same mass.

②

Define

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}, \quad \bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, \cdots \bar{\psi}_n)$$

Then,

$$\mathcal{L}_D = i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi$$

Global symmetry:

$$\Psi \rightarrow \Psi' = U\Psi \quad , (\psi'_a = U_{ab}\psi_b)$$

$$\bar{\Psi}' = \bar{\Psi}U^{+}$$

U : $n \times n$ matrix, $\partial_\mu U = 0$.

symmetry $\Rightarrow$ $U^{+}U = 1$

$\therefore$

U : Unitary, $n \times n$ matrix.

The set of all such matrices form the $U(n)$ group

$\therefore$ $\mathcal{L}_D$: invariant under global $U(1)$ transformations.

Review :

Group (def) : A group G is a set of elements $\{G_1, G_2, \cdots\}$ with a combination rule $G_1 \circ G_2$ such that

a) for any $G_1 \in G$, $G_2 \in G$,
$G_1 \circ G_2 \in G$

b) $G_1 \circ (G_2 \circ G_3) = (G_1 \circ G_2) \circ G_3$
(associativity)

c) Existence of an identity element $\mathbb{1}$:
$\mathbb{1} \circ G_1 = G_1 \circ \mathbb{1} = G_1$ for any G_1

d) Existence of inverse G_1^{-1}:
$G_1 \circ G_1^{-1} = G_1^{-1} \circ G_1 = \mathbb{1}$.

$U(n)$ is a continuous group.

$\det(U^\dagger U) = \mathbb{1} \Rightarrow |\det U|^2 = 1 \Rightarrow$

$$\det U = e^{i\varphi} \quad \text{for some } \varphi$$

But $e^{i\varphi} \in U(1)$ & $\det e^{i\varphi} = e^{i\varphi}$.
Hence in general

$$\det U = 1 \times \det U(1)$$

Consider only those U's such that

$$U^\dagger U = 1 \quad \& \quad \det U = +1$$

$\rightarrow$ "special" unitary group $SU(n)$.

Hence

$$U(n) \sim U(1) \times SU(n)$$

($\sim$ because it double counts the "center")

We already know about $U(1)$ transformations.
Let us concentrate on the $SU(n)$ part.

$\mathcal{L}_D$ has global $SU(n)$ invariance.

U: complex $n \times n$ matrix ($2n^2$ parameters)

$U^\dagger U = 1$: ($2n^2 - n^2$ parameters)

$\det U = 1$: ($2n^2 - n^2 - 1 = n^2 - 1$ parameters).

<u>Parametrization:</u>

$U \in SU(n)$, $\quad U = e^{\sum_j i\alpha_j T_j}$

α_j : parameters. $\quad T_j$: $n \times n$ matrices (fixed).

$U^\dagger = U^{-1} \Rightarrow \quad T_j = T_j^\dagger$: hermitian.

$\det U = 1 \Rightarrow \quad tr(T_j) = 0 \quad \forall j$

There are $n^2 - 1$ such matrices (count).

$$U = \exp\left(i \sum_{j=1}^{n^2-1} \alpha_j T_j\right).$$

T_j : fixed matrices : generators of the group $SU(n)$

α_j : n^2-1 parameters.

$$[T_i, T_j] = i f_{ijk} T_k \quad \text{(Lie Algebra)}$$

f_{ijk}: structure constants of the group
T_j: not uniquely determined.
α_j: n^2-1 parameters of transformation.

All choices of T_j are equivalent

Example: $SU(2)$

$$f_{ijk} = \epsilon_{ijk} \qquad (i = 1, 2, 3)$$

$$T_j = \frac{\tau_j}{2}, \quad \tau_j = \text{Pauli spin matrices.}$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{U}(\alpha_1, \alpha_2, \alpha_3) = e^{i\alpha_j \tau_j / 2}$$

Infinitesimal form: $\alpha_j \ll 1$,

$$U = \mathbb{1} + i\alpha_j T_j \qquad \left(U_{ab} = \delta_{ab} + i\alpha_j (T_j)_{ab}\right)$$

$$U^{-1} = \mathbb{1} - i\alpha_j T_j$$

$$\Psi' = (\mathbb{1} + i\alpha_j T_j)\Psi \Rightarrow \delta\Psi = i\alpha_j T_j \Psi$$

The associated conserved Noether current:

$$J^\mu_j = \bar{\Psi} \gamma^\mu T_j \Psi \equiv \bar{\psi}_a \gamma^\mu (T_j)_{ab} \psi_b$$

$$= \bar{\psi}_{a\alpha} \gamma^\mu_{\alpha\beta} (T_j)_{ab} \psi_{b\beta}$$

conserved charges:

$$Q_j = \int d^3x \left(\Psi^\dagger T_j \Psi \right) = \int d^3x \, J^0$$

Using the solution for ψ_a in terms of creation and annihilation operators, Q_j can be expressed in terms of these operators and becomes an operator acting on quantum states. However it inherits the properties of T_j and one can show that,

$$[Q_i, Q_j] = i f_{ijk} Q_k$$

Hence Q_j provide a representation of $SU(n)$ on the Hilbert space of the quantum system. Eigenstates of Q_j are states of definite $SU(n)$ charge. Since Q_j's do not all commute, these charges are not simultaneously measurable. Hence $SU(n)$ charges can be assigned with respect to a mutually commuting set of Q_j's. An interesting case is the $SU(2)$ gauge group where the charges behave like angular momenta in quantum mechanics.

Local Gauge Invariance

In the global SU(n) transformations, let

$$\alpha_j \longrightarrow g\,\theta_j(x)\,, \qquad (g:\ \text{constant})$$

$$U(x) = e^{ig\,\theta_j(x)\,T_j}$$

Then,

$$\Psi' = U(x)\,\Psi$$

But $\partial_\mu \Psi \rightarrow \partial_\mu \Psi' = \partial_\mu(U\Psi) = U(\partial_\mu \Psi) + (\partial_\mu U)\Psi$.

$\mathcal{L}_D$ is no longer invariant because of the presence of the $(\partial_\mu U)\Psi$ term.

Minimal substitution: replace $\partial_\mu \Psi$ by some $D_\mu \Psi$ such that $(D_\mu\Psi)' = U(D_\mu \Psi)$.
Then $\mathcal{L}(\Psi, D_\mu\Psi)$ will be invariant.
How to construct $D_\mu\Psi$?
Let

$$D_\mu \Psi = (\partial_\mu + ig A_\mu)\,\Psi\,.$$

Since Ψ is a column vector, A_μ is, in general, an $n\times n$ matrix. We need it to absorb unwanted SU(n) transformations, hence it should have n^2-1 degrees of freedom. One could also argue that for iD_μ to be hermitian,

we need $A_\mu = A_\mu^\dagger$, since $i\partial_\mu$ is hermitian).
Hence we write,

$$A_\mu = \sum_{j=1}^{n^2-1} A_\mu^j \, T_j$$

A_μ^j are the gauge fields and T_j the $SU(n)$ generators. A_μ is the matrix valued gauge field.

Transformation of A_μ:

Demand:

$$D'_\mu \Psi' = U D_\mu \Psi$$

$\Rightarrow$

$$(\partial_\mu + ig A'_\mu)\Psi' = (\partial_\mu + ig A'_\mu) U \Psi = U(\partial_\mu + ig A_\mu)\Psi$$

$$ig A'_\mu (U\Psi) = ig U A_\mu \Psi - (\partial_\mu U)\,\Psi$$

This is true for any Ψ. Therefore,

$$\boxed{A'_\mu = U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U) U^{-1}}$$

Using $\quad Tr(T_i T_j) = \delta_{ij}$,

$$Tr(T_i A_\mu) = A_\mu^j \, Tr(T_i T_j) = A_\mu^i .$$

Now, the gauge invariant Lagrangian is

$$\mathcal{L} = i \bar{\Psi} \gamma^\mu D_\mu \Psi - m \bar{\Psi} \Psi$$

or $\mathcal{L} = i\bar{\Psi}\not{D}\Psi - m\bar{\Psi}\Psi - g J^{\mu}_{j} A^{j}_{\mu}$

where $J^{\mu}_{j} = \bar{\Psi}\gamma^{\mu} T_j \Psi$.

This is a generalization of interaction between photons and charged fermions. The A^j_μ are the analogues of photon, but differing in a crucial way: they are not neutral and also carry SU(N) charges:

What is the analogue of Maxwell Lagrangian and $F_{\mu\nu}$?

consider $D_\mu D_\nu \Psi$.

It is easy to show that

$$D'_\mu D'_\nu \Psi' = U(D_\mu D_\nu \Psi)$$

consider $D_\mu D_\nu \Psi - D_\nu D_\mu \Psi \equiv [D_\mu, D_\nu]\Psi$.

Now,

$$D_\mu D_\nu \Psi = (\partial_\mu + igA_\mu)(\partial_\nu \Psi + igA_\nu \Psi)$$

$$= \partial_\mu\partial_\nu \Psi + ig\partial_\mu A_\nu \Psi + igA_\nu \partial_\mu \Psi + igA_\mu \partial_\nu \Psi - g^2 A_\mu A_\nu \Psi.$$

$\Rightarrow$

$$[D_\mu, D_\nu]\Psi = ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\right)\Psi.$$

Define:

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]}$$

This is the non-Abelian generalization of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in Maxwell theory ($\partial_\mu A_\nu - \partial_\nu A_\mu$ alone does not have nice $SU(N)$ transformation properties)

$$F_{\mu\nu} = F^i_{\mu\nu} T_i$$

$$F^i_{\mu\nu} = \mathrm{Tr}(T_i F_{\mu\nu}) = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + ig f_{ijk} A^j_\mu A^k_\nu$$

Transformation of $F_{\mu\nu}$:

$$[D_\mu, D_\nu]\,\Psi = ig F_{\mu\nu} \Psi .$$

$$[D'_\mu, D'_\nu]\,\Psi' = ig F'_{\mu\nu} \Psi'$$

$$\Rightarrow \quad U([D_\mu, D_\nu]\,\Psi) = ig F'_{\mu\nu} U \Psi$$

$$\Rightarrow \quad \boxed{F'_{\mu\nu} = U F_{\mu\nu} U^{-1}.}$$

Gauge invariant Lagrangian for A^i_μ:

$$\frac{1}{4}\,\mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right) = \frac{1}{4}\,\mathrm{Tr}\left(F'_{\mu\nu} F'^{\mu\nu}\right) \quad (\text{gauge invariant})$$

Note that:

$$\mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right) = F^i_{\mu\nu} F^{\mu\nu j}\, \mathrm{Tr}(T_i T_j) = \sum_{i=1}^{N^2-1} F^i_{\mu\nu} F^{\mu\nu i} .$$

The gauge field Lagrangian (Yang-Mills theory)

$$\mathcal{L}_{YM} = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} = -\frac{1}{4} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$$

A^i_μ : non-Abelian gauge fields, or Yang-Mills fields.

Beside quadratic terms $\mathrm{Tr}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$, $\mathcal{L}_{YM}$ also contains cubic terms,

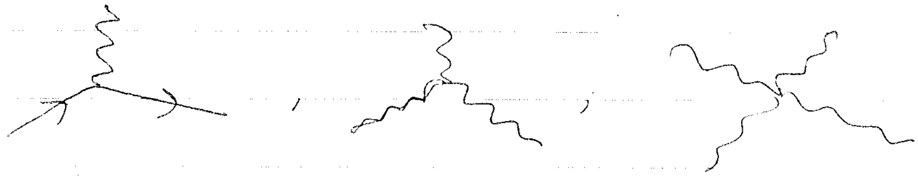
$$\mathrm{Tr}\left(ig\,(\partial_\mu A_\nu - \partial_\nu A_\mu)[A^\mu, A^\nu]\right)$$

and quartic terms,

$$\mathrm{Tr}\,(-g^2)\left([A_\mu, A_\nu][A^\mu, A^\nu]\right).$$

Hence, in contrast to photons, A^i_μ are self-interacting.

Thus in the Yang-Mills theory, coupled to fermions, we have interaction vertices of the type



The quadratic (free) theory is quantized after gauge fixing. This is more involved than QED due to the non-Abelian nature and often involves introducing extra (ghost fields). The cubic & quartic terms are then treated as interactions.

An Application: QCD

Quarks are the building blocks of hadrons. The pauli exclusion principle and experimental data indicate that every quark comes in three varieties (called colors) which are experimentally not distinguishable from each other. Hence rotations in the color space should be a symmetry of the theory. If "i" is a color index, then

$$\psi_i \to U_{ij}\psi_j \qquad (i=1,2,3)$$

with $U^\dagger U = 1$ should be a symmetry of the Dirac Lagrangian for the colored quarks.
$\to$ global $SU(3)$ invariance.

To keep $\mathcal{L}_D$ invariant under the corresponding local transformations, we need to introduce $n^2-1=8$ gauge fields. These are the "gluon" fields that mediate strong interactions among quarks. Gluon self-interaction causes the interactions to grow at low energies (confinement) and become weak at high energies (asymptotic freedom). To describe the hadron spectrum, we need 6 types of quarks (up, down, charm, strange, bottom, top). Each of these comes in 3 colors. Quarks also have weak & electromagnetic interactions not included in QCD.

Chiral Fermions

Dirac spinor ψ_α $(\alpha = 1,2,3,4)$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Define: $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$(\gamma_5)^2 = \mathbb{1}$, γ_5 is real, $\mathrm{Tr}(\gamma_5) = 0$.

$\Rightarrow$ the eigenvalues of γ_5 are $+1, +1, -1, -1$.

In a diagonal basis, $\gamma_5 = \begin{pmatrix} +1 & & & 0 \\ & +1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

Eigenvectors corresponding to these eigenvalues can be easily constructed:

Introduce

$$P_L = \frac{1}{2}(1-\gamma_5), \qquad P_R = \frac{1}{2}(1+\gamma_5)$$

$P_L^2 = P_L$, $P_R^2 = P_R$, $P_L + P_R = 1$, $P_L R_R = 0$

$\Rightarrow$ P_L & P_R are projection operators

Define:

$$\psi_L = P_L \psi = \frac{1}{2}(1-\gamma_5)\psi$$

$$\psi_R = P_R \psi = \frac{1}{2}(1+\gamma_5)\psi$$

$$\psi = \psi_L + \psi_R .$$

If ψ has 4 independent components, ψ_L & ψ_R will each have 2.

$$\gamma_5 \psi_L = -\psi_L$$

$$\gamma_5 \psi_R = +\psi_R$$

ψ_L, ψ_R: chiral spinors.

L : left
R : right } the reason for the names will be clarified later.

since $\{\gamma^\mu, \gamma_5\} = 0$,

$$\gamma_5 (\gamma^\mu \psi_L) = (\gamma^\mu \psi_L)$$

$$\gamma_5 (\gamma^\mu \psi_R) = -(\gamma^\mu \psi_R)$$

Note that,

$$i \bar{\psi} \not{\partial} \psi = i (\bar{\psi}_L \not{\partial} \psi_L + \bar{\psi}_R \not{\partial} \psi_R)$$

$$m \bar{\psi} \psi = m (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)$$

$\Rightarrow$ Both ψ_L & ψ_R are needed to write a mass term (Dirac mass term).

$\boxed{\begin{array}{l} \psi_L \\ \psi_R \end{array} : \text{irreducible representations of the Lorentz group.} \\ \psi = \psi_L + \psi_R : \text{reducible rep of the Lorentz group}}$

what is the physical significance of ψ_L & ψ_R?

ψ_L :	chirality	-1	for any mass m.
ψ_R :	chirality	$+1$	

Helicity :

We defined fermion helicity as $\sigma_p = \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$.

It measures the component of spin in the direction of motion.

$\vec{\sigma} = \{\sigma^1, \sigma^2, \sigma^3\}$: (4x4 matrices).

where,

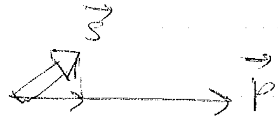
$\sigma^1 = \sigma^{23}, \ \sigma^2 = \sigma^{31}, \ \sigma^3 = \sigma^{12}$

&

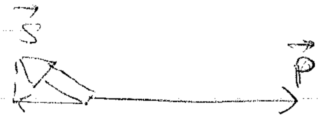
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$: generators of the Lorentz group acting on spinors ($\mu, \nu = 1, 2, 3$: space rotations).

Fermions have two definite helicity states:

$\langle \sigma_p \rangle = +\frac{1}{2}\hbar$



$\langle \sigma_p \rangle = -\frac{1}{2}\hbar$



In ψ, this helicity information is encoded in the spinors $u_r(\vec{p})$ and $v_r(\vec{p})$, $(r=1,2)$. Recall that u_r and $\bar{u}_r$ are associated with electrons ($\bar{u}_r$ with creation & u_r with annihilation) and v_r & $\bar{v}_r$ are associated with positrons (v_r with creation & $\bar{v}_r$ with annihilation).

$$\psi = \sum_{\vec{p},r} \left(\frac{m}{VE_p}\right)^{1/2} \left[c_r(\vec{p})\, u_r(\vec{p})\, e^{-ipx} + d_r^\dagger(\vec{p})\, v_r(\vec{p})\, e^{ipx} \right]$$

$$\bar{\psi} = \sum_{\vec{p},r} \left(\frac{m}{VE_p}\right)^{1/2} \left[d_r(\vec{p})\, \bar{v}_r(\vec{p})\, e^{-ipx} + c_r^\dagger(\vec{p})\, \bar{u}_r(\vec{p})\, e^{ipx} \right]$$

$$\sigma_p\, u_r(\vec{p}) = (-1)^{r+1} u_r(\vec{p}) \qquad \sigma_p\, v_r(\vec{p}) = (-1)^{r} v_r(\vec{p})$$

$\Rightarrow$

$r=1$: $c_1^\dagger|0\rangle$, $d_1^\dagger|0\rangle$: +ve helicity 1-particle state

$r=2$: $c_2^\dagger|0\rangle$, $d_2^\dagger|0\rangle$: −ve helicity 1-particle state.

note that u_r appears with c_r but v_r appears with $d_r^\dagger$, hence the lack of symmetry between u_r and v_r

Helicity projection operators :

$$\Pi^{\pm}(\vec{p}) = \frac{1}{2}(1 \pm \sigma_p).$$

$$\Pi^{+}(\vec{p})\, u_r(\vec{p}) = \delta_{1r}\, u_r(\vec{p}) \qquad \Pi^{+} v_r = \delta_{2r}\, v_r$$

$$\Pi^{-}(\vec{p})\, u_r(\vec{p}) = \delta_{2r}\, u_r(\vec{p}) \qquad \Pi^{-} v_r = \delta_{1r}\, v_r$$

For $m=0$, chirality coincides with helicity.

i.e.,

let $\quad w_r(\vec{p}) = $ either $u_r(\vec{p})$ or $v_r(\vec{p})$

Then

$$\gamma_5\, w_r(\vec{p}) = \sigma_p\, w_r(\vec{p}).$$

(This is easy to prove, see Appendix A of Mandl & Shaw).

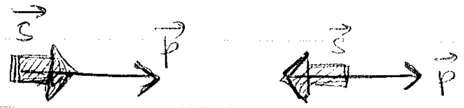
Hence,

$$\Pi^{\pm}(\vec{p}) = \frac{1}{2}(1 \pm \gamma_5)$$

Even for $m \neq 0$, at high energies m can be neglected and the correspondance holds.

Hence, for $m=0$ (or small m)

$$\Psi = \Psi_L + \Psi_R$$



QED and QCD do not distinguish between the two helicity states of fermions. But weak interactions are sensitive to chirality. (Parity violation – Lee & Yang (theory), Wu (expt)). (β - decay of ^{60}Co, etc)