Symmetries & Conservation laws

Noether's theorem (statement):

consider a field theory action

$$S = \int_\Omega d^4x \, \mathcal{L}\left(\phi_a(x), \partial_\mu \phi_a(x)\right)$$

Ω denotes a finite region of space-time. consider infinitesimal transformations of coordinates and fields,

$$\boxed{\begin{aligned} x^\mu \longrightarrow x'^\mu &= x^\mu + \delta_\epsilon x^\mu \\ \phi_a(x) \longrightarrow \phi'_a(x') &= \phi_a(x) + \delta_\epsilon \phi(x) \end{aligned}}$$

(✱) The change in $\phi_a(x)$ could be partly due to $\delta_\epsilon x^\mu$ and partly due to some intrinsic transformation of the fields that will not vanish even when $\delta_\epsilon x^\mu = 0$.

(✱) The subscript ϵ in δ_ϵ indicates that the transformations may depend on a number of infinitesimal parameters collectively denoted by ϵ. The transformations are linear in these parameters and we only retain terms linear in the ϵ in various expansions.

(*) It is very useful to introduce another type of variation $\bar{\delta}_\epsilon \phi$ as:

$$\boxed{\bar{\delta}_\epsilon \phi_a(x) = \phi'_a(x) - \phi_a(x).}$$

Note that $\bar{\delta}_\epsilon \phi$ is calculated with both ϕ' and ϕ having the same argument x.

The Noether theorem states that under such transformations, the variation of the action is given by,

$$\delta S = \int_\Omega d^4x \left(\partial_\mu J^\mu_\epsilon \right)$$

with

$$J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta}_\epsilon \phi_a + \mathcal{L}\, \delta x^\mu$$

Invariance of S ($\delta S = 0$) for any Ω leads to the current conservation eqn.

$$\boxed{\partial_\mu J^\mu_\epsilon = 0}$$

This in turn allows us to define a conserved charge,

$$\boxed{Q_\epsilon = \int_V d^3x \; J^0_\epsilon}$$

The conservation of Q_ϵ follows from

$$\frac{\partial Q_\epsilon}{\partial x^0} = \int_V d^3x \, \partial_0 J^0_\epsilon = -\int_V d^3x \, \vec{\nabla} \cdot \vec{J}_\epsilon = -\oint\!\!\oint_{\partial V} d\vec{S} \cdot \vec{J}_\epsilon$$

Thus the variation of Q_ϵ within V is caused by a flux of the charge across the boundary of V. When V includes the entire space, then $\partial Q_\epsilon / \partial x^0 = 0$.

Noether theorem (Proof):

Under transformations $x^\mu \to x'^\mu$ and $\phi_a(x) \to \phi'_a(x')$, the variation of the action is given by:

$$\delta S = \int_{\Omega'} d^4x' \, \mathcal{L}\left(\phi'(x'), \partial'_\mu \phi'(x')\right) - \int_{\Omega} d^4x \, \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right)$$

$$= \int_{\Omega'} d^4x \, \mathcal{L}\left(\phi'(x), \partial_\mu \phi'(x)\right) - \int_{\Omega} d^4x \, \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right)$$

where we have used the fact that x' is a dummy integration variable. Now, using,

$$\phi'_a(x) = \phi_a(x) + \bar{\delta}_\epsilon \phi_a, \quad \text{one has}$$

$$\mathcal{L}\left(\phi'(x), \partial_\mu \phi'(x)\right) = \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) + \frac{\partial \mathcal{L}}{\partial \phi_a} \bar{\delta}_\epsilon \phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta}_\epsilon(\partial_\mu \phi_a)$$

$$= \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) + \frac{\partial \mathcal{L}}{\partial \phi_a} \bar{\delta}_\epsilon \phi_a + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \bar{\delta}_\epsilon \phi_a\right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}\right) \bar{\delta}_\epsilon \phi_a$$

$= 0$ (by eqn. of motion)

$\therefore$

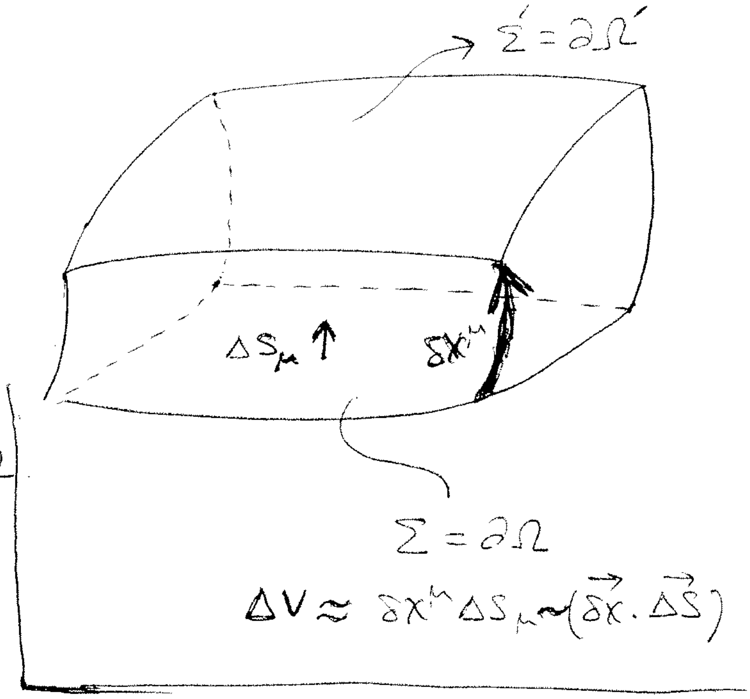
$$\delta S = \int_{\Omega'} d^4x \, \mathcal{L} - \int_{\Omega} d^4x \, \mathcal{L} + \int_{\Omega'} d^4x \, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta}_\epsilon \phi_a\right)$$

$$= \int_{(\Omega'-\Omega)} d^4x \, \mathcal{L} + \int_{\Omega} d^4x \, \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a} \bar{\delta}_\epsilon \phi_a\right) + \overset{\approx 0}{O(\delta x \, \bar{\delta} \phi)}$$

$\Omega' - \Omega$ is the volume bounded between the boundaries of Ω' and Ω. Since the boundary of Ω is shifted to the boundary of Ω' by a coordinate shift δx^μ, we have,

$$\Omega' - \Omega = \int_\Sigma dS_\mu \, \delta_\epsilon x^\mu$$

where $\{dS_\mu\}$ denotes an area element on the boundary Σ of Ω.



Thus, comparing with

$$\Omega' - \Omega = \int_{\Omega' - \Omega} d^4x = \int_\Sigma dS_\mu \, \delta_\epsilon x^\mu,$$

we have,

$$\int_{\Omega' - \Omega} d^4x \, \mathcal{L} = \int_\Sigma dS_\mu (\delta_\epsilon x^\mu \mathcal{L}) = \int_\Omega d^4x \, \partial_\mu (\mathcal{L} \, \delta_\epsilon x^\mu)$$

where, the last step follows from the divergence theorem. Hence,

$$\boxed{\begin{aligned} \delta S &= \int_\Omega d^4x \; \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta}_\epsilon \phi_a + \mathcal{L} \, \delta_\epsilon x^\mu \right) \\ &= \int_\Omega d^4x \left(\partial_\mu J^\mu_\epsilon \right) \end{aligned}}$$

from which the Noether theorem stated earlier follows.

Some Examples

$$J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \bar{\delta}_\epsilon \varphi_a + \mathcal{L}\,\delta x^\mu.$$

It is more useful to rewrite this in terms of $\delta_\epsilon \varphi_a$:

$$\delta_\epsilon \varphi_a = \varphi_a'(x') - \varphi_a(x) = \varphi_a'(x + \delta x) - \varphi_a(x)$$

$$= \varphi_a'(x) + \delta_\epsilon x^\mu \partial_\mu \varphi_a(x) - \varphi_a(x) = \bar{\delta}_\epsilon \varphi_a(x) + \delta_\epsilon x^\mu \partial_\mu \varphi_a(x).$$

or,

$$\boxed{\bar{\delta}_\epsilon \varphi_a(x) = \delta_\epsilon \varphi_a(x) - \delta_\epsilon x^\mu \partial_\mu \varphi_a(x).}$$

Therefore,

$$J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta_\epsilon \varphi_a - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \partial_\nu \varphi_a - \delta^\mu_\nu \mathcal{L} \right) \delta x^\nu$$

Using,

$$\boxed{T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \partial_\nu \varphi_a - \delta^\mu_\nu \mathcal{L}},$$

we have,

$$\boxed{J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta_\epsilon \varphi_a - T^\mu{}_\nu \delta x^\nu}$$

$T^\mu{}_\nu$ is called the stress-energy tensor, or the energy-momentum tensor the significance of which will become clear below.

The associated conserved quantity is

$$\int_V d^3x \; J^0_\epsilon = \int d^3x \left\{ \pi^a \delta_\epsilon \varphi_a - (\pi^a \partial_\nu \varphi - \mathcal{L}\delta^0_\nu)\delta_\epsilon x^\nu \right\}$$

Now, we consider some specific transformations and the associated conserved quantities.

<u>Global Gauge Transformations:</u>

The simplest such transformations act on the complex scalar and Dirac spinor as

$$\delta x^\mu = 0, \qquad \phi'(x) = e^{iq\epsilon}\phi(x), \quad \phi^{*\prime}(x) = e^{-iq\epsilon}\phi^*(x).$$

and,

$$\delta x^\mu = 0, \qquad \psi'_\alpha(x) = e^{iq\epsilon}\psi_\alpha(x), \quad \bar{\psi}'_\alpha(x) = e^{-iq\epsilon}\bar{\psi}_\alpha(x).$$

For $\partial_\mu \epsilon = 0$ (global transformations) it is easy to check that the corresponding Lagrangians are invariant. q is a fixed constant. For infinitesimal ϵ,

$$\delta\phi = iq\epsilon\phi, \quad \delta\phi^* = -iq\epsilon\phi^*$$

and,

$$\delta\psi_\alpha = iq\epsilon\psi_\alpha, \quad \delta\bar{\psi}_\alpha = -iq\epsilon\bar{\psi}_\alpha .$$

For (ϕ, ϕ^*),

$$J^\mu_\epsilon = \epsilon(iq)\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\phi - \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^*}\phi^*\right) = \epsilon J^\mu.$$

For ψ_α,

$$J^\mu_\epsilon = \epsilon(iq)\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_\alpha}\psi_\alpha = \epsilon J^\mu$$

The corresponding conserved quantity could be electric charge, lepton no., etc. To identify the nature of the conserved charge we also need to know how the transformation acts on other fields in the theory. The transformation considered here is a $U(1)$ transformation. In general one can also have $U(n)$ or $SU(n)$ transformations acting on the fields, e.g.,

$$\delta\phi_a = i\,\epsilon_A T^A_{ab}\phi_b$$

which is the infinitesimal form of

$$\phi'(x) = e^{i\epsilon_A T^A}\phi(x) \quad , \quad \phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{pmatrix}$$

The $n\times n$ matrices T^A are called the generators of transformation and satisfy,

$$[T^A, T^B] = i f^{AB}{}_C T^C.$$

These lead to "Non-Abelian" conserved charges. Later in the course we come across gauge transformations for which $\partial_\mu \epsilon_A \neq 0$ i.e., local gauge transformations.

Space-time translations :

These act as

$$x^\mu \longrightarrow x'^\mu = x^\mu + \epsilon^\mu \quad \Rightarrow \quad \delta x^\mu = \epsilon^\mu$$

where ϵ^μ are 4 constant parameters. One can easily compute how fields transform: Under a general coordinate transformation, scalars and vectors transform as,

$$\phi'(x') = \phi(x) \quad \Rightarrow \quad \boxed{\delta\phi = 0}$$

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) = \frac{\partial(x'^\nu - \delta x^\nu)}{\partial x'^\mu} A_\nu(x)$$

$$= \left(\delta^\nu_\mu - \frac{\partial(\delta x^\nu)}{\partial x'^\mu}\right) A_\nu(x).$$

$$\Rightarrow \quad \boxed{\delta A_\mu = A'_\mu(x) - A_\mu(x) = -\frac{\partial(\delta x^\nu)}{\partial x'^\mu} A_\nu(x)}$$

For rigid translations, $\delta x^\nu = \epsilon^\nu$ = constant, so that $\partial(\delta x^\nu)/\partial x'^\mu = 0 \Rightarrow \delta A_\mu = 0$. Spinors are also invariant under general coordinate transformations,

$$\psi'_\alpha(x') = \psi_\alpha(x) \quad \Rightarrow \quad \delta\psi_\alpha = 0.$$

Therefore, for translations,

$$\boxed{\delta_\epsilon \phi = 0, \quad \delta_\epsilon A_\mu = 0, \quad \delta_\epsilon \psi_\alpha = 0}$$

or generically,

$$\boxed{\delta\phi_a = 0}$$

<u>Caution</u>: We have stated that spinors are invariant under "general coordinate transformations". Later, we will see that they transform under rotations. This is because, strictly speaking, rotations do not act on the "coordinate space" but rather on the "tangent space" to the coordinate space. The difference is obvious in curved spaces but in flat space, the tangent space coincides with the space itself and masks the difference between transformation of the space and its tangent space.

The associated conservation laws are contained in:

$$J^\mu_\epsilon = - T^\mu{}_\nu \, \epsilon^\nu$$

Since ϵ^ν are all independent, we infact have 4 conserved currents. Taking out the parameter $(-\epsilon^\nu)$,

$$J^\mu_\epsilon = -T^\mu{}_\nu \, \epsilon^\nu = -\epsilon^\nu J^\mu{}_\nu .$$

The 4 <u>conserved currents</u> are,

time translation: $J^\mu{}_0 = T^\mu{}_0$

space translations: $J^\mu{}_i = T^\mu{}_i \quad (i = 1, 2, 3).$

The current conservation eqn. implies

$$\boxed{\partial_\mu T^{\mu\nu} = 0} \qquad \left(T^{\mu\nu} = T^\mu{}_\rho \, \eta^{\rho\nu} \right)$$

The associated conserved "charges" are (put $\mu = 0$)

time translation: $\boxed{H = \int d^3x \, T^0{}_0 = \int d^3x \left(\pi^a \dot{\phi}_a - \mathcal{L}\right)}$

space translations: $\boxed{P_i = \int d^3x \, T^0{}_i = \int d^3x \left(\pi^a \partial_i \phi_a\right)}$

Clearly, **H** is the hamiltonian. **P_i** are the momenta of the fields (because they are the conserved quantities associated with spacial translations). In this sense we have been able to define the notion of <u>Energy</u> and <u>Momentum</u> for fields.

One can also introduce energy and momentum densities,

$$\mathcal{H} = \pi^a \dot{\phi}_a - \mathcal{L}, \qquad \mathcal{P}_i = \pi^a \partial_i \phi_a$$

Lorentz Transformations (Including rotations):

Lorentz transformation are given as $x'^\mu = L^\mu{}_\nu x^\nu$ where $L^T \eta L = \eta$. For infinitesimal transformations, $L^\mu{}_\nu$ can be expanded as $L^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$. Then the condition on $L^\mu{}_\nu$ implies that $\omega_{\mu\nu} = -\omega_{\nu\mu}$ $\left(\omega_{\mu\nu} = \eta_{\mu\rho}\omega^\rho{}_\nu\right)$.
Thus, under an infinitesimal Lorentz transformation,

$$x'^\mu - x^\mu = \delta x^\mu = \omega^\mu{}_\nu x^\nu \quad (\text{to lowest order in } \omega)$$

or

$$x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu, \quad \text{or} \quad x^\mu = x'^\mu - \omega^\mu{}_\nu x'^\nu.$$

How do Lorentz transformations act on the fields?

Scalars: $\phi'(x') = \phi(x) \Rightarrow \delta_\epsilon \phi = 0$.

vectors: we have obtained the transformation of A_μ under an infinitesimal general coordinate transformation as $\delta A_\mu = -\partial(\delta x^\nu)/\partial x'^\mu \, A_\nu(x)$.
For rotations, $\delta x^\nu = \omega^\nu{}_\mu x^\mu = \omega^\nu{}_\mu x'^\mu$ to first order in $\omega^\mu{}_\nu$. Hence, $\partial(\delta x^\nu)/\partial x'^\mu = \omega^\nu{}_\mu$ and

$$\boxed{\delta A_\mu = -\omega^\nu{}_\mu A_\nu = \omega_\mu{}^\nu A_\nu}$$

Spinors: We have stated that general coordinate transformations (including Lorentz rotations in the above sence) do not affect the spinors. This is still true. However in flat space any rotation of the coordinate space must be accompanied by a similar rotation of the tangent space. This is so because, by convention, coordinate space and tangent space basis vectors are taken to be parallel to each other. If we rotate one and not the other, then we change this convention to which spinors are sensitive. This is the reason why rotations also affect spinors while translations do not. The effect of a rotation on the spinor is given by

$$\boxed{\delta \psi_\alpha = \omega_{\rho\sigma} S^{\rho\sigma}_{\alpha\beta} \psi_\beta} \qquad (\rho, \sigma = 0, 1, 2, 3)$$

$S^{\rho\sigma} = -S^{\sigma\rho}$ are six 4×4 matrices (for $(\rho\sigma) = (1,2), (1,3), (2,3), (1,0), (2,0), (3,0)$) with components $(S^{\rho\sigma})_{\alpha\beta}$ $\alpha, \beta = 1, \cdots 4$.

They are the generators of the Lorentz group in the spinor representation.

Remark: In general the Lorentz group $SO(1,3)$ could act on any ϕ_a, where "a" stands for a vector index "μ" or a spinor index "α" or a pair of indices "$\mu\nu$", etc. In the generic case,

$$\delta \phi_a = \frac{1}{2} \omega_{\rho\sigma} \bar{S}^{\rho\sigma}_{ab} \phi^b$$

where, now $\bar{S}^{\rho\sigma}$ are the generators of the Lorentz group in appropriate representation. For spinors, $\bar{S}^{\rho\sigma}_{ab}$ becomes $S^{\rho\sigma}_{\alpha\beta}$ and is given in terms of the Dirac matrices $\gamma^{\mu}_{\alpha\beta}$:

$$\boxed{S^{\rho\sigma} = \frac{1}{4}[\gamma^{\rho}, \gamma^{\sigma}]}$$

For vectors, ϕ_a is replaced by A_μ and $\bar{S}^{\rho\sigma}_{ab}$ by $M^{\rho\sigma}_{\mu\nu}$. These are infact 6 matrices $M^{\rho\sigma}$ (i.e, M^{12}, M^{13}, M^{23}, M^{01}, M^{02}, M^{03}), with components $(M^{\rho\sigma})_{\mu\nu}$ (This may look a little confusing since the matrices and their components are labled by the same set of indices). By this account the transformation of vector fields should take the form

$$\boxed{\delta A_\mu = \frac{1}{2} \omega_{\rho\sigma} (M^{\rho\sigma})_{\mu}{}^{\nu} A_\nu}$$

This indeed is consistent with the transformations we have derived simply because $M^{\rho\sigma}$ are such that

$$\frac{1}{2}\,\omega_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu} = \omega_{\mu\nu}$$

Explicitly, we have $(M^{\rho\sigma})_{\mu\nu} = \delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \delta^{\rho}_{\nu}\delta^{\sigma}_{\mu}$. These are the generators of $SO(1,3)$ in the vector representation.

(End of remark)

To summarize, under Lorentz transformations,

$$\delta\phi_a = \frac{1}{2}\omega_{\rho\sigma}\,\bar{S}^{\rho\sigma}_{ab}\,\phi^b$$

where, for scalars, $\bar{S} = 0$, for vectors $\bar{S}^{\rho\sigma}_{ab}$ is $M^{\rho\sigma}_{\mu\nu}$ with $M^{\rho\sigma}_{\mu\nu} = \delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \delta^{\rho}_{\nu}\delta^{\sigma}_{\mu}$ and for spinors, $S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho,\gamma^\sigma]$ and the γ^μ can be constructed in terms of the Pauli spin matrices $\sigma^1,\sigma^2,\sigma^3$.

The conserved quantities are contained in

$$J^\mu_\omega = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\left(\frac{1}{2}\omega_{\rho\sigma}\bar{S}^{\rho\sigma}_{ab}\phi^b\right) - T^\mu{}_\nu\,\omega^\nu{}_\sigma\,x^\sigma$$

$$= \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{S}^{\rho\sigma}_{ab}\phi^b\omega_{\rho\sigma} - \frac{1}{2}\left(T^{\mu\rho}x^\sigma - T^{\mu\sigma}x^\rho\right)\omega_{\rho\sigma}$$

$$= \left[\frac{1}{2}\left(T^{\mu\sigma}x^\rho - T^{\mu\rho}x^\sigma\right) + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bar{S}^{\rho\sigma}_{ab}\phi^b\right]\omega_{\rho\sigma}.$$

Removing the arbitrary transformation parameters, $\frac{\omega_{\rho\sigma}}{2}$, one defines the 6 conserved currents $(\bar{\mathcal{M}}^\mu)^{\sigma\rho}$,

$$\boxed{\bar{M}^{\mu\rho\sigma} = \left(T^{\mu\sigma}x^{\rho} - T^{\mu\rho}x^{\sigma}\right) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\bar{S}^{\rho\sigma}_{ab}\varphi^b}$$

The corresponding conserved charges are

$$\boxed{\bar{M}^{\rho\sigma} = \int d^3x\, \mathcal{M}^{0\rho\sigma} = \int d^3x\, \left(T^{0\sigma}x^{\rho} - T^{0\rho}x^{\sigma}\right) + \pi^a \bar{S}^{\rho\sigma}_{ab}\varphi^b}$$

Ordinary space rotations correspond to $\omega^{0i} = 0$ and are parametrized by $\omega^{ij} = -\omega^{ji}$. The corresponding conserved quantities $\bar{M}^{ij}$ are the conserved angular momentum components, (using $T^{0i} = p^i$),

$$\boxed{\bar{M}^{ij} = \int d^3x \left[\left(x^i p^j - x^j p^i\right) + \pi^a \bar{S}^{ij}_{ab}\varphi^b\right]}$$

$$\left(M^{12} = J^3,\ M^{31} = J^2,\ M^{23} = J^1\right)$$

Remembering that p^i was the momentum density along x^i, the first term clearly is the orbital angular momentum. The second term is independent of the origin of the coordinate system is corresponds to the spin angular momentum. It depends entirely on the transformation properties of the field under $SO(3)$ (contained in $\bar{S}^{ij}_{ab}$).

Now we have constructed all the quantities needed to characterize particles in terms of field variable. Thus, we can associate similar properties to the states on which the quantum fields $\hat{\Phi}_a$ act.

Exercise: (a) Compute the $U(1)$ charges for the complex scalar & Dirac (spinor) fields.

(b) compute H, P_i, M^{ij} (only the spin part) for real scalars, complex scalars, vector fields and Dirac fields.

(c) Express the conserved quantities computed above in terms of the creation and annihilation operators for the corresponding fields.