# Review of Free Field Quantización

## Review ①: Quantización of Discrete Systems

Dynamical variables: $q_i(t)$, $(i = 1, 2, \cdots n)$

Lagrangian : $L = L(q_i(t), \dot{q}_i(t))$

Action :

$$\boxed{S[q(t)] = \int_{t_1}^{t_2} dt \, L(q_i(t), \dot{q}_i(t))}$$

Equation of motion (least action principle):

Let $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$, such that $\delta q_i(t)\Big|_{t_1, t_2} = 0$

Demand $\delta S = 0$ under the variation. This picks up only those $q_i(t)$ that extremize the action:

$$\begin{aligned}
\delta S &= S[q + \delta q] - S[q] \\
&= \int_{t_1}^{t_2} dt \left( L(q + \delta q, \dot{q} + \delta \dot{q}) - L(q, \dot{q}) \right) \\
&= \int_{t_1}^{t_2} dt \left( \cancel{L(q, \dot{q})} + \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i - \cancel{L(q, \dot{q})} \right) \\
&= \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_i} \delta q_i + \cancelto{0}{\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right)} - \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right) \\
&= - \int_{t_1}^{t_2} dt \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \right) \delta q_i(t) .
\end{aligned}$$

$$\delta S = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0}$$

Momentum canonically conjugate to $q_i$ :

$$\boxed{p^i = \frac{\partial L}{\partial \dot{q}_i}}$$

Hamiltonian: $$\boxed{H = \sum_{i=1}^{n} p^i \dot{q}_i - L}$$

Note that $H = H(q_i, p_i)$

The Poisson bracket:

For $F = F(q_i, p^i)$ , $G = G(q_i, p^i)$,

$$\boxed{\{F, G\}_{P.B.} \equiv \sum_{i=1}^{n} \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial G}{\partial q_i} \right)}$$

Equation of motion in the Hamiltonian formulation:

$$\dot{q}_i(t) = \frac{\partial H}{\partial p^i} \quad , \quad \dot{p}^i(t) = -\frac{\partial H}{\partial q_i}$$

or,

$$\dot{q}_i = \{q_i, H\}_{P.B.} \quad , \quad \dot{p}^i = \{p^i, H\}.$$

For a general $F = F(p^i(t), q_i(t), t)$

$$\boxed{\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}_{P.B.}}$$

In particular, $\boxed{\{q_i, p^j\}_{P.B.} = \delta^i_j}$ (same time)

③

Quantization:

Introduce operators: $q_i \rightarrow \hat{q}_i \; , \; p^i \rightarrow \hat{p}^i$

Replace: $\{q_i, p^j\}_{PB} \longrightarrow \frac{1}{i\hbar}[\hat{q}_i, \hat{p}^j]$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

Then, $\boxed{[\hat{q}_i, \hat{p}^j] = i\hbar\, \delta^j_i}$

( we do not yet consider constrained systems)

Now, one studies the quantum theory in the Schrodinger or Heisenberg formulations.

Bound systems: energy levels, wave functions, etc.
Unbound systems: scattering amplitudes, etc.

Basic feature: the number and identity of particles taking part in the quantum process is conserved.

This is a major limitation!

# Systems with continuous degrees of freedom

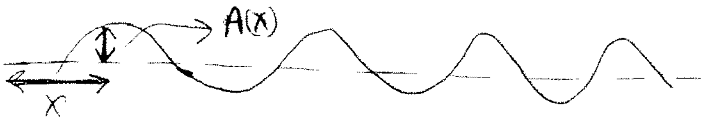
$$q_i(t) \longrightarrow q(t, \vec{x})$$

( the discrete variable $i = 1, \dots, n$ becomes the continuous space coordinate $\vec{x}$ ).

Examples :

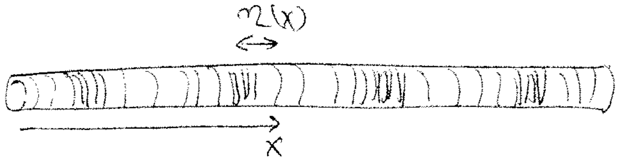
ⓐ Transverse oscillations of a string :

$A(\vec{x}, t)$



ⓑ Longitudinal waves in an elastic rod :

$\eta(\vec{x})$ : displacement about average position at distance $x$.



ⓒ Sound waves : $P(\vec{x})$ ( pressure variation)

ⓓ Dynamics of fluids : $\vec{v}(\vec{x})$ ( velocity field governed by Navier-Stokes equation)

(All the above are continuous limits of discrete mechanical systems and the continuum mechanics can be derived discrete mechanics)

ⓔ Electromagnetic fields. $A_\mu(\vec{x}, t)$ ( vector potential). ( Dynamics : Maxwell theory)

ⓕ Schrodinger wave function $\psi(\vec{x}, t)$.
(Dynamics : Schrodinger Eqn.)

( ⓔ & ⓕ have no discrete analogues.
ⓔ arises in the classical theory but not ⓕ.)

It turns out that all these diverse continuous systems can be described in a unified way using the frame work of mechanics of continuous media, or so called, classical field theory.

## Review ②: Classical Free Field Theory:

Fields: $\{\phi_a(\vec{x}, t)\} \equiv \{\phi_a(x)\}$

$(a = 1, 2, \ldots \quad)$

⊛ scalar fields. $\phi = \phi^*$ (real), $\phi, \phi^*$ (complex)
(Higgs)

⊛ vector fields: $A_\mu(x)$ ( electromagnetism, weak & strong interactions)

⊛ Dirac (spinor) fields: $\psi_\alpha(x)$ ( matter: leptons & quarks).

⊛ metric field: $g_{\mu\nu}(x)$ ( gravity )

## Lagrangian Density:

$$\boxed{\mathcal{L} = \mathcal{L}\left(\phi_a(x),\ \partial_\mu \phi_a(x)\right)} \quad , \quad \left(\partial_\mu \phi_a = \frac{\partial \phi_a}{\partial x^\mu}, \quad \mu = 0, 1, \cdots 3\right)$$

**Action:** $$S[\phi(x)] = \int_\Omega d^4x\, \mathcal{L}\left(\phi_a, \partial_\mu \phi_a\right)$$

$d^4x = dx^0\, d^3\vec{x}$.

$\Omega$ : 4-dim. volume.

## Eqn. of motion:

$$\phi_a(x) \longrightarrow \phi_a(x) + \delta\phi_a(x)$$

such that $\delta\phi_a(x)\Big|_{\partial\Omega} = 0$.

Demand $\delta S = 0 \Rightarrow$ configurations $\phi_a(x)$ for which $S[\phi]$ is stationary (Maxma or Minima).

$$\delta S = S[\phi + \delta\phi,\ \partial_\mu\phi + \partial_\mu\delta\phi] - S[\phi]$$

$$= \int_\Omega d^4x \left(\mathcal{L}(\phi + \delta\phi,\ \partial_\mu\phi + \partial_\mu\delta\phi) - \mathcal{L}(\phi, \partial_\mu\phi)\right)$$

$$= \int_\Omega d^4x \quad \mathcal{L}(\cancel{\phi, \partial_\mu\phi}) + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) - \mathcal{L}(\cancel{\phi, \partial_\mu\phi})$$

$$= \int_\Omega d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\right)\delta\phi_a\right)$$

Here we have used $\delta(\partial_\mu \phi) = \partial_\mu \delta\phi$.

Now, by the 4-dim generalization of the divergence theorem,

$$\int_\Omega d^4x \; \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi\right) = \int_{\partial\Omega} dS_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) = 0$$

since $\delta\phi\big|_{\partial\Omega} = 0$

( $d^4x$: 4-dim volume element

$dS_\mu$: 3-dim surface element $\left(dS_\mu = \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\, dx^\nu dx^\rho dx^\sigma\right)$ )

$\therefore \quad \delta S = 0 \Rightarrow$

$$\boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0}$$

( The Euler-Lagrange Eqn )

<u>Discussion in the class</u>: i) How can one understand using $\delta\phi$ variations in the least action principle? These are more sofisticated generalizations of the notion of "virtual displacement" used to solve problems in statics ( see, for examples, Sommerfeld's book on classical mechanics).

ii) Can one understand the action $S$ and the least action principle in physical terms?

In quantum mechanics the action is related to the phase of the wave function. The least action principle follows as the characterization of the most probable path in the path integral formulation.

The momentum canonically conjugate to $\phi_a(x)$ is defined as

$$\pi^a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}$$

where $\dot{\phi}_a = \frac{d}{dt}\phi_a = c\frac{d}{dx^0}\phi_a$ (if c=1 convention is not used).

Hamiltonian density:

$$\mathcal{H} = \sum_a \pi^a \dot{\phi}_a - \mathcal{L}$$

This Legendre transform converts the $\dot{\phi}_a$ dependence of $\mathcal{L}$ into a $\pi^a$ dependence of $\mathcal{H}$. i.e.

$$\mathcal{H} = \mathcal{H}(\phi_a, \pi^a; \partial_i \phi_a)$$

The Hamiltonian is

$$H = \int d^3x \, \mathcal{H} .$$

There is a Poisson bracket formulation of the Hamiltonian theory that we will not discuss here (we will come back to this later) except to say that,

$$\{\phi_a(x), \pi^b(y)\}_{\substack{PB \\ (x^0=y^0)}} = \delta_a^{\;b}\, \delta^{(3)}(\vec{x}-\vec{y}) .$$

## Path to Quantization.

Using the standard prescription, to quantize the system, one promotes $\phi_a$ and $\pi^a$ to operators and replaces the Poisson braket by a commutator:

$$\phi_a(x) \longrightarrow \hat{\phi}_a(x), \qquad \pi^b(x) \longrightarrow \hat{\pi}^b(x)$$

$$\left[\hat{\phi}_a(x), \hat{\pi}^b(y)\right]\Big|_{x^0=y^0} = i\,\delta_a^{\ b}\,\delta^3(\vec{x}-\vec{y}).$$

$$\left[\hat{\phi}_a(x), \hat{\phi}_b(y)\right]\Big|_{x^0=y^0} = 0 = \left[\hat{\pi}^a(x), \hat{\pi}^b(y)\right]\Big|_{x^0=y^0}$$

The quantum theory follows from the above Equal Time commutation Relations (ETCRs).

Remarks:

1) what about the commutators for $x^0 \neq y^0$? These can be derived from the ETCR's and need not be imposed as postulates.

2) For spin 1/2 fields (spinor fields) the commutator is replaced by an anti-commutator (this is demanded by causality and leads to the Pauli exclusion principle for fermions).

3) The appearance of $\delta^3(\vec{x}-\vec{y})$ in the ETCR's follows from special relativity: since any information needs a finite travel time between $\vec{x}$ and $\vec{y}$, for $\vec{x} \neq \vec{y}$, for $x^0-y^0=0$, $\hat{\phi}_a(x)$ and $\hat{\pi}^a(y)$ cannot affect each other unless $\vec{x}-\vec{y}=0$.

## How to deal with the quantized theory

A field theory is quantized by imposing ETCR's on the fields. How to proceed further from here? Since $\hat{\phi}$ and $\hat{\pi}$ are now operators, they act on some space of states (the Hilbert space, or the Fock space), mapping some states to some other, say, $|s'\rangle = \hat{\phi}|s\rangle$.

Now one has to address two problems:

a) How to construct the states on which $\hat{\phi}$ & $\hat{\pi}$ act?

b) How to associate a physical meaning to them in terms of particles?

a) The space of states can be constructed in terms of a set of basis states (usually, using creation and annihilation operators). To carry this out explicitly, we have to consider specified field theories and solve the equations of motion (in the free field case).

b) From ordinary quantum theory we know that states on which quantum operators act represent physical systems. One can get some idea about the correspondance between states and the physical systems they represent, by considering the classical limit of the theory. However, for some quantum field theories, the physical system may not even have the usual classical limit. In this case the correspondance can established by studying the transformation properties of fields and the associated conserved quantities.

The claim is that the states on which $\hat{\phi}(x)$ act represent systems of elementary particle. Note that elementary particles are characterized by a set of quantum numbers, or conserved quantities, like mass m ( $m^2 = E^2 - \vec{p}^2$ ), spin, electric charge, lepton number, color, etc. These, in turn are conserved quantities associated with certain symmetry transformations. On the other hand, these transformations can be naturally defined for the fields and as a result, one can associate the same types of conserved quantities to the fields. This allows us to establish connections between quantum fields and elementary particles.

In the following, we will first review some free field theories that we are interested in, and then review the Noether theorem that establishes the connection symmetries and conservation law.

# Some Free Field Theory Examples

## ① The Real scalar field:

$$\phi(x) = \phi^{+}(x).$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right) \Rightarrow (\Box + m^2)\phi = 0$$

$(\Box = \partial_\mu\partial^\mu)$.

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi(x).$$

$$H = \int d^3x \;\; \frac{1}{2}\left(\dot\phi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \geqslant 0$$

Quantization:

ETCR's : $\left.[\hat\phi(x), \hat{\dot\phi}(y)]\right|_{x^0=y^0} = i\,\delta^{(3)}(\vec{x}-\vec{y})$ ,

with other $[\,,\,] = 0$.

## ② The complex scalar field:

$$\phi(x) \;,\; \phi^{+}(x).$$

$\mathcal{L} = \partial_\mu\phi^{*}\partial^\mu\phi - m^2\phi^{+}\phi$ (Treat $\phi$ & $\phi^{+}$ as independent fields).

$$(\Box + m^2)\,\phi(x) = 0, \qquad (\Box + m^2)\,\phi^{+}(x) = 0.$$

$$\pi(x) = \dot\phi^{*}(x) \;,\; \pi^{*}(x) = \dot\phi(x)$$

Quantization: $\phi \rightarrow \hat{\phi}$, $\phi^* \longrightarrow \hat{\phi}^\dagger$,

$$[\phi(x), \dot{\phi}^\dagger(y)]\Big|_{x^0=y^0} = i\,\delta^{(3)}(\vec{x}-\vec{y}) \,.$$

## ③ The Spinor (Dirac) Field

This is a 4-component field, $\psi_\alpha(x)$, $\alpha = 1, 2, 3, 4$.

The values $\alpha = 1, \cdots, 4$ DONOT correspond to the space-time coordinates. The 4-components can be arranged into a column vector, $\psi(x)$,

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

The theory also contains 4 matrices $\gamma^\mu$, $\mu = 0, 1, 2, 3$ satisfying the Dirac algebra,

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}.$$

Each $\gamma^\mu$ can be represented by a $4\times4$ matrix with elements $\gamma^\mu_{\alpha\beta}$, $(\alpha, \beta = 1, \cdots, 4)$. It is useful to choose a representation for $\gamma^\mu$'s where,

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^i \qquad (i = 1, 2, 3).$$

Both these properties are combined into

$$\boxed{\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0}$$

(Many other representations are possible. We make a choice based on convenience).

Adjoint of $\psi(x)$: $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \qquad \psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$$

$\gamma^0$: 4×4 matrix.

(In the quantum theory $\psi_\alpha^* \longrightarrow \psi_\alpha^\dagger$).

Thus, in components,

$$\bar{\psi}_\alpha = \sum_{\beta=1}^{4} \psi_\beta^\dagger \gamma^0_{\beta\alpha} .$$

Lagrangian:

$$\mathcal{L} = \bar{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x)$$

$$= (\cdots\cdots)\begin{pmatrix} \vdots & \cdots & \cdots \\ \vdots & \cdots & \cdots \\ \vdots & \cdots & \cdots \end{pmatrix}\begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

$$= \sum_{\alpha,\beta=1}^{4} \bar{\psi}_\alpha(x)\left(i\gamma^\mu_{\alpha\beta}\partial_\mu - m\,\delta_{\alpha\beta}\right)\psi_\beta(x)$$

$(\hbar = c = 1)$

Eqns. of motion: $i\gamma^\mu \partial_\mu \psi - m\psi = 0$

$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$

The Feynman "slash" notation:

$$\gamma^\mu \partial_\mu = \not{\partial} \qquad (\gamma^\mu C_\mu = \not{C}).$$

Then,

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi,$$

$$(i\not{\partial} - m)\Psi = 0, \qquad i\partial_\mu \bar{\Psi}\gamma^\mu + m\bar{\Psi} = 0.$$

Note that in the above, $m$ actually stands for $m\mathbb{1}_{4\times4}$.

Conjugate momenta:

$$\pi^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}_\alpha} = i\Psi_\alpha^\dagger, \qquad \bar{\pi}_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\Psi}}_\alpha} = 0.$$

$$H = \int d^3x\, \mathcal{H} = \int d^3x \quad \bar{\Psi}(-i\gamma^i\partial_i + m)\Psi$$

Quantization:

$$\left[\Psi_\alpha(x), \Psi_\beta^\dagger(y)\right]_+\Big|_{x^0=y^0} = \delta_{\alpha\beta}\,\delta^{(3)}(\vec{x}-\vec{y}).$$

where, $[A,B]_+ = AB + BA$: anti-commutator. This is the only possibility to get a consistent quantum theory. Since $\bar{\Psi}_\lambda = \sum_\beta \Psi_\beta^\dagger \gamma^0_{\beta\lambda}$,

$$\left[\Psi_\alpha(x), \bar{\Psi}_\lambda(y)\right]_+\Big|_{x^0=y^0} = \gamma^0_{\alpha\lambda}\,\delta^{(3)}(\vec{x}-\vec{y}).$$

## ⑦ Vector fields:

$A_\mu(x)$, $\mu = 0, \dots, 3$. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Eqn. of motion:

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0$$

( Maxwell's eqns in the absence of sources.

Gauge invariance:

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda \Rightarrow F'_{\mu\nu} = F_{\mu\nu}$$

Hence

$$\mathcal{L}(A') = \mathcal{L}(A).$$

This allows us to choose $\partial^\nu A_\nu = 0$ (Lorenz gauge) so that

$$\boxed{\Box A^\mu = 0}$$

(Eqn. of motion in the Lorenz gauge).

Conjugate momenta:

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{\mu 0}.$$

$$\pi^i = -F^{i0} \neq 0$$

$$\pi^0 = -F^{00} = 0 \quad \text{(Problem!)}$$

( No momentum conjugate $A_0$, How to quantize $A_0$ ?)

Reason: of the 4-components of $A_\mu$, only 2 are physical. (One can be removed by using the freedom in the choice of the gauge transformation parameter $\Lambda(x)$ and the other can be eliminated using the eqn. of motion and a residual gauge invariance, $A_\mu \to A_\mu + \partial_\mu \tilde{\Lambda}$, such that $\Box \tilde{\Lambda} = 0$).

We have to take this into account before quantizing the theory.

One approach is to start with the modified Lagrangian,

$$\mathcal{L} = -\frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu).$$

Leading to $\Box A^\mu = 0$.

Hence this is equivalent to the original theory provided $\partial_\mu A^\mu = 0$. Now,

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -\dot{A}^\mu \neq 0$$

Quantization:

$$[A_\mu(x), \pi^\nu(y)]\Big|_{x^0=y^0} = -[A_\mu(x), \dot{A}^\nu(y)]\Big|_{x^0=y^0} = i\delta_\mu^\nu \, \delta^{(3)}(\vec{x}-\vec{y})$$

One has to make sure that the constraint $\partial_\mu A^\mu = 0$ is also implemented in the quantum theory in an appropriate way. This is acheived by the so called, Gupta-Bleuler quantization in a simple way. This is a simple example of a system with a constraint.

We will see that the vector field described above corresponds to massless particles of spin 1. A mass term in the Lagrangian has the form,

$$\frac{1}{2} m^2 A_\mu A^\mu .$$

Massive spin 1 fields (or particles) are needed to explain weak interactions. However, a mass term $m^2 A_\mu A^\mu$ does not preserve gauge invariance and does not lead to a consistent theory (problem with renormalizability). Hence mass terms for vector fields should be introduced in a much more subtle way. This will be the subject of spontaneous symmetry breaking and Higgs mechanism.

Now we turn to a discussion of Noether's theorem. Our intention is to find expressions for the Energy, momentum, angular momentum, spin and charges of the fields. This will enable us to interpret the content of quantum theory in terms of elementary particles.