Conserved Charges as Generators of Transformations:

Noether's theorem shows that the invariance of the action under a transformation leads to the existance of a conserved charge Q_ϵ. However, there is more to the relation between Q_ϵ and the corresponding transformation. Here, we show that Q_ϵ acts as the generator of the transformation in the sense that

$$\bar{\delta}_\epsilon \phi_a(x) = -\left\{ Q_\epsilon, \phi_a(x) \right\}_{P.B.}$$

where $\{\cdot,\cdot\}_{P.B.}$ stands for the "Poisson bracket" and $\bar{\delta}_\epsilon \phi_a$ was defined in our discussion of Noether's theorem. But first, let us make a digression and discuss some concepts that will be needed for the proof:

Digression

Ⓐ Functional Derivatives: Let $f_a(y)$ be functions in an n-dimensional space-time with coordinates y^α, $(\alpha = 0, 1, \ldots n-1)$. Consider a function $\mathcal{F}$ of the f_a and their first derivatives, $\mathcal{F} = \mathcal{F}\left(f_a(y), \partial_\alpha f_a(y)\right)$ Then

$$F[f_a] = \int_\Omega d^n y \; \mathcal{F}\left(f_a(y), \partial_\alpha f_a(y)\right)$$

is said to be a "functional" of f_a (it depends on the functional form of $f_a(y)$). Consider a local variation of $f_a(y)$:

$$f_a(y) \longrightarrow f_a(y) + \delta f_a(y)$$

By "local" we mean that $\delta f_a(y)$ vanishes on the boundary $\partial\Omega$ of the integration region Ω. Then the variation of F is

$$\delta F = F[f_a + \delta f_a] - F[f_a] = \int_\Omega d^n y \left(\frac{\partial \mathcal{F}}{\partial f_a} \delta f_a + \frac{\partial \mathcal{F}}{\partial(\partial_\alpha f_a)} \delta(\partial_\alpha f_a) \right)$$

where we have Taylor expanded $\mathcal{F}$ in δf_a and $\delta(\partial_\alpha f_a)$ retaining the lowest terms for an infinitesimal variation. Using $\delta(\partial_\alpha f_a) = \partial_\alpha(\delta f_a)$, the second term can be integrated by parts giving,

$$\delta F = \int_\Omega d^n y \left(\frac{\partial \mathcal{F}}{\partial f_a} - \partial_\alpha \left(\frac{\partial \mathcal{F}}{\partial(\partial_\alpha f_a)} \right) \right) \delta f_a$$

where, we have used $\delta f_a|_{\partial\Omega}$ to drop a boundary term. The functional derivative of F is then given by

$$\boxed{\frac{\delta F}{\delta f_c(y)} = \frac{\partial \mathcal{F}(y)}{\partial f_c(y)} - \partial_\alpha \frac{\partial \mathcal{F}(y)}{\partial(\partial_\alpha f_c(y))}}$$

In this notation, the variation δF becomes,

$$\delta F = \int_\Omega d^n y \, \frac{\delta F}{\delta f_c(y)} \delta f_c(y)$$

(B) Poisson brackets in Field Theory:

Consider a theory of fields ϕ_a. The canonically conjugate momenta are given by

$$\pi^a(y) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)}$$

Let F and G be functionals of ϕ_a and π^a. Then their Poisson bracket is defined as

$$\{F, G\}_{PB} = \int_\Sigma d^{n-1}y \sum_c \left(\frac{\delta F}{\delta \phi_c(y)} \frac{\delta G}{\delta \pi^c(y)} - \frac{\delta F}{\delta \pi^c(y)} \frac{\delta G}{\delta \phi_c(y)} \right)$$

The derivatives above are functional derivatives and the integral is over the space part of Ω.

(c) Hamiltonian formulation of classical field theory

The Hamiltonian density $\mathcal{H}$ and the Hamiltonian H are defined as

$$\mathcal{H} = \pi^a \partial_0 \phi_a - \mathcal{L}, \qquad H = \int_\Sigma d^{n-1}y \, \mathcal{H}$$

Under a variation of the fields,

$$\delta\mathcal{H} = \delta\pi^a \partial_0 \phi_a + \pi^a \delta(\partial_0 \phi_a) - \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a - \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi_a} \delta(\partial_\alpha \phi_a)$$

$$= \delta\pi^a \partial_0 \phi_a - \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a - \frac{\partial \mathcal{L}}{\partial \partial_i \phi_a} \delta(\partial_i \phi_a)$$

where $i = 1, \ldots n-1$. Thus the independent variables in $\mathcal{H}$ are π^a, ϕ_a and $\partial_i \phi_a$.

$$\mathcal{H} \equiv \mathcal{H}(\pi^a, \phi_a, \partial_i \phi_a)$$

so that,

$$\delta\mathcal{H} = \frac{\partial \mathcal{H}}{\partial \pi^a} \delta\pi^a + \frac{\partial \mathcal{H}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{H}}{\partial(\partial_i \phi_a)} \delta(\partial_i \phi_a)$$

Comparing the two expressions for $\delta\mathcal{H}$ we get

$$\frac{\partial\mathcal{H}}{\partial\pi^a} = \dot{\phi}_a \quad , \quad \frac{\partial\mathcal{H}}{\partial(\partial_i\phi_a)} = -\frac{\partial\mathcal{L}}{\partial(\partial_i\phi_a)} \quad , \quad \frac{\partial\mathcal{H}}{\partial\phi_a} = -\frac{\partial\mathcal{L}}{\partial\phi_a}$$

The third equation above can be rewritten using the Euler-Lagrange equations as

$$\frac{\partial\mathcal{H}}{\partial\phi_a} = -\frac{\partial\mathcal{L}}{\partial\phi_a} = -\sum_{\alpha=0}^{n-1}\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi_a)} = -\partial_0\frac{\partial\mathcal{L}}{\partial\partial_0\phi_a} - \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\phi_a)}$$

$$= -\dot{\pi}^a + \partial_i\frac{\partial\mathcal{H}}{\partial(\partial_i\phi_a)}$$

Thus we have the two Hamilton's equations for the fields ϕ_a:

$$\dot{\phi}_a = \frac{\partial\mathcal{H}}{\partial\pi^a}$$

$$\dot{\pi}^a = -\left(\frac{\partial\mathcal{H}}{\partial\phi_a} - \partial_i\frac{\partial\mathcal{H}}{\partial\partial_i\phi_a}\right)$$

Remembering that $H = \int_\Sigma d^{n-1}y\ \mathcal{H}(\phi_a, \pi_a, \partial_i\phi_a)$ and using the definition of functional derivatives, these can be reexpressed in a compact form as

$$\dot{\phi}_a = \frac{\delta H}{\delta\pi^a} \quad , \quad \dot{\pi}^a = -\frac{\delta H}{\delta\phi_a}$$

These can also be recast in terms of Poisson brackets if we note that

$$\frac{\delta \phi_a(y)}{\delta \phi_b(y')} = \delta_a^b \, \delta^{n-1}(y - y') .$$

Then, using the definition of the Poisson bracket given above, we can see that

$$\boxed{\begin{aligned} \dot{\phi}_a &= \{ \phi_a , H \}_{P.B.} \\ \dot{\pi}^a &= \{ \pi^a , H \}_{P.B.} \end{aligned}}$$

This completes our digression.

Conserved Charges as Generators of Transformations

From Noether's theorem, the conserved charge is given by

$$Q_\epsilon = \int d^{n-1}y \, J^0 = \int d^{n-1}y \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \bar{\delta}_\epsilon \phi_a + \mathcal{L} \, \delta_\epsilon y^0 \right)$$

where

$$\bar{\delta}_\epsilon \phi_a = \tilde{\phi}_a(y) - \phi_a(y) = \left(\tilde{\phi}_a(\tilde{y}) - \phi_a(y) \right) - \left(\tilde{\phi}_a(\tilde{y}) - \tilde{\phi}_a(y) \right)$$

$$= \delta_\epsilon \phi_a - \partial_\beta \phi \, \delta_\epsilon y^\beta$$

In general we deal with transformations where $\delta_\epsilon \phi_a$ is a function of the ϕ_a but not their derivatives. Using this fact, we can express Q_ϵ in terms of ϕ_a, π^a and $\partial_i \phi_a$ so that its Poisson brackets can be easily

computed:

$$Q_\epsilon = \int d^{n-1}y \left(\left(\pi^a (\delta_\epsilon \phi_a - \partial_\beta \phi_a \delta_\epsilon y^\beta) + (\pi^a \partial_0 \phi_a - \mathcal{H}) \delta_\epsilon y^0 \right) \right.$$

$$= \int d^{n-1}y \quad \pi^a (\delta_\epsilon \phi_a - \partial_i \phi_a \delta_\epsilon y^i) - \int d^{n-1}y \left(\mathcal{H} \delta_\epsilon y^0 \right)$$

$$\left\{ Q_\epsilon , \phi_b(y) \right\}_{P.B} = \sum_c \int d^{n-1}y' \left(\frac{\delta Q_\epsilon}{\delta \phi_c(y')} \frac{\delta \phi_b(y)}{\delta \pi_c(y')} - \frac{\delta Q_\epsilon}{\delta \pi_c(y')} \frac{\delta \phi_b(y)}{\delta \phi_c(y')} \right)$$

$$= - \frac{\delta Q_\epsilon}{\delta \pi^b(y)}$$

$$= -(\delta_\epsilon \phi_b - \partial_i \phi_b \delta_\epsilon y^i - \dot{\phi}_b \delta_\epsilon y^0)$$

$$= -(\delta_\epsilon \phi_b - \partial_\alpha \phi_b \delta_\epsilon y^\alpha) = - \bar{\delta}_\epsilon \phi_b(y).$$

Hence we have shown that

$$\boxed{\bar{\delta} \phi_b(y) = - \left\{ Q_\epsilon , \phi_b(y) \right\}_{P.B}}$$

In the quantum theory, Poisson brackets give way to commutators,

$$\boxed{\bar{\delta} \phi_b(y) = i \left[Q_\epsilon , \phi_b(y) \right]}$$