QFT-I problem set 1

(Fawad Hassan)

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1. (a) Show that the inhomogeneous Maxwell equations $\partial_\mu F^{\mu\nu} = J^\nu$ are the equations of motion corresponding to the Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

   (b) Show that the corresponding Hamiltonian is given by (after setting $J^\mu = 0$)

$$H = \int d^3x (\frac{1}{2}E^2 + \frac{1}{2}B^2)$$

   where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields

2. (a) Consider the theory of a real scalar field $\phi(x)$. Starting from the general expression for conserved charges as given by Noether's theorem, obtain the expressions for the conserved charges $\vec{P}$ and H associated with invariance under space and time translations, respectively.

   (b) Using the expansion of $\phi(x)$ in terms of creation and annihilation operators, show that

$$\vec{P} = \sum_{\vec{k}} \hbar \vec{k} a_k^\dagger a_k \qquad H = \sum_{\vec{k}} \hbar \omega_k a_k^\dagger a_k$$

3. Problem 3.1 from Mandl and Shaw

4. Given a state $|n_k >= \frac{1}{\sqrt{n_k!}}(a_k^\dagger)^{n_k}|0>$, show that $< n_k|n_k >= 1$ (It is convenient to first evaluate the commutator $[a_k, (a_k^\dagger)^{n_k}]$).

5. For a complex scalar field, starting from the general expression for conserved charges given by Noether's theorem, calculate the conserved charge Q associated with the global gauge transformations $\phi \to e^{iq\theta}\phi$. Using the expansion of the fields in terms of creation and annihilation operators show that

$$Q = q\sum_{\vec{k}}(a_k^\dagger a_k - b_k^\dagger b_k)$$

6. Problem 3.4 from Mandl and Shaw

7. According to Noether's theorem, invariance of the action under the trasformations $x^\mu \to x^\mu + \delta x^\mu$ and $\phi_a(x) \to \tilde{\phi}_a(\tilde{x})) = \phi_a(x) + \delta\phi_a(x)$ leades to the conserved current

$$J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\bar{\delta}\phi_a + \mathcal{L}\delta x^\mu$$

where $\bar{\delta}\phi_a = \delta\phi_a - \partial_\nu\phi_a\delta x^\nu$. Show that the corresponding conserved charge can be written as

$$Q_\epsilon = \int d^3x(\pi^a(\delta\phi_a - \partial_i\phi_a\delta x^i) - \mathcal{H}\delta x^0)$$

and that it serves as the generator of the transformation, *i.e.*,

$$\bar{\delta}\phi_b(x) = -\{Q_\epsilon\,,\,\phi_b(x)\}_{PB}$$

Here $\mathcal{H}$ is the Hamiltonian density and the Poisson bracket of, say, f and g is defined as

$$\{f, g\}_{PB} = \sum_c \int d^3z \left(\frac{\delta f}{\delta\phi_c(z)}\frac{\delta g}{\delta\pi^c(z)} - \frac{\delta g}{\delta\phi_c(z)}\frac{\delta f}{\delta\pi^c(z)}\right)$$

δ denotes *functional* differentiation though that is not important here.