Path-Integral Formulation of Quantum Mechanics

Problem set II

Based on R. Shankar, Sections 8.3 and 8.4

(Fawad Hassan, Jan 2003)

1. For a free particle moving in one dimension, consider a path from x_0 at time t_0 to x_N at time t_N in the $x-t$ plane. Divide the time interval $t_N - t_0$ into N segments of length $\epsilon = \frac{t_N - t_0}{N}$. For large enough N, the continuous path $x(t)$ can be approximated by the discretized path, $x_n = x(t_0 + n\epsilon)$, for $n = 0, 1, \cdots, N$. Now discretize the free particle action
$$S = \int_{t_0}^{t_N} \frac{1}{2} m\dot{x}^2 dt$$

2. Evaluate the multiple integral,
$$U_\epsilon(x_N, t_N; x_0, t_0) = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \cdots \int_{-\infty}^{+\infty} dx_{N-1} \exp\left[\frac{i}{\hbar}\frac{m}{2}\sum_{i=0}^{N-1}\frac{(x_{i+1}-x_i)^2}{\epsilon}\right]$$
<u>Comment:</u> *This is a path integral over the family of discretized paths. We are finally interested in the path integral over the family of all continuous paths connecting the two end points.*

3. The path integral
$$U(x_N, t_N; x_0, t_0) = \int \mathcal{D}[x(t)]\, e^{iS/\hbar}$$
can be operationally defined as
$$U(x_N, t_N; x_0, t_0) = \lim_{\substack{N\to\infty \\ \epsilon \to 0 \\ N\epsilon \to t_N - t_0}} A_{\epsilon,N}\, U_\epsilon(x_N, t_N; x_0, t_0)$$
The normalization $A_{\epsilon,N}$ is needed to render the limit well-defined. This normalization can be fixed by using the requirement that as $(x_N - x_0) \to 0$, $U(x_N, t_N; x_0, t_0) \to \delta(x_N - x_0)$. Determine $A_{\epsilon,N}$ and evaluate $U(x_N, t_N; x_0, t_0)$, using the delta-function representation,
$$\delta(x - x') = \lim_{\Delta\to 0} \frac{1}{\sqrt{\pi\Delta^2}} \exp\left[-\frac{(x-x')^2}{\Delta^2}\right]$$
<u>Comment:</u> *The physical meaning of the quantity $U(x_N, t_N; x_0, t_0)$ which is computed by the path integral will be clarified later*