Path-Integral Formulation of Quantum Mechanics

Problem set I

(Fawad Hassan, Jan 2003)

1. Consider a particle in one dimension moving in a potential $V(x)$ with instantaneous velocity $v = \dot{x} \equiv dx/dt$. Write down the (non-relativistic) equation of motion using Newton's second law.

2. The difference between the kinetic energy and potential energy is called the *Lagrangian* funtion L. For the above system,

$$L = \frac{1}{2}mv^2 - V(x)$$

Now, evaluate

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

regarding x and $\dot{x}$ as independent variables. Compare with the answer of question no 1.
Comment: *The equatin of motion can always be written in terms of the Lagrangian as above*

3. For a *free* particle, $V(x) = 0$. Solve the equation of motion (that is, find $x(t)$) such that at time t_1 the particle is at position x_1 and at time t_2 it is at position x_2. Draw an $x - t$ plot of the path.
Comment: *A path $x(t)$ that solves the equation of motion is called the "classical" path*

4. Consider a path $x = At^2 + Bt$, where the constant A is a free parameter and B is determined by the requirement that $x(t=0) = 0$ and $x(t=T) = X \quad (T > 0)$.

   (a) How does this path look on an $x - t$ plot?

   (b) Is a free particle allowed to follow this path classically? Find the value of A corresponding to the classically allowed path.

5. The *Action* is defined as

$$S = \int_{t_1}^{t_2} L(x, \dot{x})dt$$

   (a) Evaluate the action for a free particle along its classical path (for simplicity, take $x_1 = 0$, $t_1 = 0$).

(b) Evaluate the free particle action for the path $x = At^2 + Bt$ in question number 4. Different values of A correspond to different paths. Find the path for which the action S has its minimum value.

Comment: *The value of the action S in general depends on the path along which it is evaluated. The Classical paths always correspond to minima (and sometimes the maxima) of the action. The action evaluated along a classical path is called the "classical action".*

6. Show that

$$\int_{-\infty}^{+\infty} dy\, e^{-a\,y^2} = \sqrt{\frac{\pi}{a}}, \qquad (\text{for } \mathrm{Re}\, a > 0)$$

Also, evaluate

$$\int_{-\infty}^{+\infty} dy\, e^{-a\,y^2 + b\,y}$$

and

$$\int_{-\infty}^{+\infty} dy\, y^2\, e^{-a\,y^2}$$

Comment: *Although this result is easily obtained for* $\mathrm{Re}\, a > 0$, *it can also be applied to the case of purely imaginary a.*

7. In question number 4, we considered a family of paths $x(t) = At^2 + Bt$ parameterized by A and connecting $x_1 = 0$ and $x_2 = X$. In 5(b), we obtained the free particle action $S(A)$ over this family of paths. Now, evaluate

$$\int_{A_1}^{A_2} dA\, e^{iS(A)/\hbar}$$

Comment: *This is a simple example of a path-integral, where the quantity $e^{iS(A)/\hbar}$ is integrated over the family of paths parameterized by A. Of course, there are infinitely many more paths connecting x_1 and x_2 which are not accounted for by this parameterization. To include all paths, one needs an infinite number of parameters, for example, as in $x(t) = \sum_{i>1} A_i t^i + Bt$. The path-integral over all these paths will then involve an infinite dimensional integral over all the parameters A_i, i.e., $\int dA_1 dA_2 dA_3 \cdots$.*