Electrodynamics I – Solutions to the extra problems

By *Emil Johansson Bergholtz, ejb@physto.se*

These solutions are preliminary and there may very likely exist some typos. If you find any I would be appreciate if you let me know.

Extra problem no. 1

This problem is solved by expanding the solution of the Laplace equation in terms of orthonormal functions. In this particular example it is fruitful to expand the potential in terms of a Legendre series according Jackson eq.(3.33) since the problem has azimuthal symmetry. Hence we write the potential as

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}[A_\ell r^\ell + B_\ell r^{-(\ell+1)}]P_\ell(\cos\theta), \tag{1}$$

where A_ℓ and B_ℓ are real constants that we are to determine from the boundary conditions. First of all we note that the potential have to stay finite everywhere. Therefore we decompose the solution into separate cases and write

$$\Phi(r,\theta) = \Phi_{in}(r,\theta) = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta), \quad r < R \tag{2}$$

$$\Phi(r,\theta) = \Phi_{out}(r,\theta) = \sum_{\ell=0}^{\infty} b_\ell r^{-(\ell+1)} P_\ell(\cos\theta), \quad r > R. \tag{3}$$

The potential must of course be continuous at $r = R$ thus $\Phi_{in}(R,\theta) = \Phi_{out}(R,\theta)$. Plugging this condition into eqs. (2) and (3) yields $a_\ell = b_\ell R^{-(2\ell+1)}$. Furthermore we make use of the general condition for the surface charge $\sigma = \epsilon_0 \frac{\partial\Phi}{\partial n}$, where $\frac{\partial}{\partial n}$ denotes the normal derivative. Since we know that $\sigma = \sigma_0 \cos\theta = \sigma_0 P_1(\cos\theta)$ we obtain

$$\sigma_0 P_1(\cos\theta) = \epsilon_0\left(\frac{\partial\Phi_{in}}{\partial r} - \frac{\partial\Phi_{out}}{\partial r}\right)_{r=R}$$

$$= \epsilon_0 \sum_{\ell=0}^{\infty}[\ell a_\ell R^{\ell-1} + (\ell+1)b_\ell R^{-(\ell+2)}]P_\ell(\cos\theta) = \epsilon_0 \sum_{\ell=0}^{\infty}(2\ell+1)a_\ell R^{\ell-1}P_\ell(\cos\theta). \tag{4}$$

The equation above is solved by $a_1 = \frac{\sigma_0}{3\epsilon_0}$ and $a_\ell = 0$ for all $\ell \neq 1$. Hence we have arrived at the exact expression for the potential both inside and outside the shell of the sphere:

$$\Phi_{in}(r,\theta) = \frac{\sigma_0}{3\epsilon_0} r\cos\theta, \quad r < R \tag{5}$$

$$\Phi_{out}(r,\theta) = \frac{\sigma_0 R^3}{3\epsilon_0 r^2}\cos\theta, \quad r > R. \tag{6}$$

The electric field follows directly from $\overrightarrow{E} = -\nabla\Phi$:

$$\overrightarrow{E}_{in} = -\frac{\sigma_0}{3\epsilon_0}\hat{z}, \quad r < R \tag{7}$$

$$\overrightarrow{E}_{out} = \frac{\sigma_0 R^3}{3\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}), \quad r > R. \tag{8}$$

Extra problem no. 2

By use of the identity

$$\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} = -\nabla(\frac{1}{|\mathbf{x}-\mathbf{x}'|}), \tag{9}$$

we find that

$$\begin{aligned} \mathbf{B} &= \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x' \\ &= -\frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}') \times \nabla(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) d^3x' \\ &= \frac{\mu_0}{4\pi}\nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x', \end{aligned} \tag{10}$$

where the last step follows from $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$ and the fact that J does not depend on $\mathbf{x}$. This immediately yields Gilberts law, $\nabla \cdot \mathbf{B} = 0$, since the divergence of a curl of a vector always vanishes.

To proceed we take the curl of (10) and use the identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$:

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}\nabla \int \mathbf{J}(\mathbf{x}') \cdot \nabla(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) d^3x' - \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{x}')\nabla^2(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) d^3x' \tag{11}$$

The second of these integrals is easily taken care of since $\nabla^2(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) = -4\pi\delta(\mathbf{x}-\mathbf{x}')$. Hence it yields a term $\mu_0\mathbf{J}(\mathbf{x})$. The first integral needs a bit more care. By use of a partial integration, the continuity equation, the definition of the potential $\phi(\mathbf{x})$ and a bit of ingenuity we find

$$\begin{aligned} \frac{\mu_0}{4\pi}\nabla \int \mathbf{J}(\mathbf{x}') \cdot \nabla(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) d^3x' &= -\frac{\mu_0}{4\pi}\nabla \int \mathbf{J}(\mathbf{x}') \cdot \nabla'(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) d^3x \\ = \frac{\mu_0}{4\pi}\nabla \int \frac{\nabla' \cdot \mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x' &= -\frac{\epsilon_0}{4\pi}\nabla \int \frac{\frac{\partial\rho(\mathbf{x}')}{\partial t}}{|\mathbf{x}-\mathbf{x}'|} d^3x' \\ &= -\frac{\mu_0}{4\pi}4\pi\epsilon_0\frac{\partial}{\partial t}\nabla\phi(\mathbf{x}) = \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{12}$$

Thus we have arrived at the Maxwell-Ampere law

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}. \tag{13}$$

Extra problem no. 3

Since $\mathbf{E}_0$ is a constant vector it follows that

$$\nabla \cdot \mathbf{E}(\mathbf{x},t) = (E_0^x \frac{\partial}{\partial x} + E_0^y \frac{\partial}{\partial y} + E_0^z \frac{\partial}{\partial z}) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$
$$= i(E_0^x k_x + E_0^y k_y + E_0^z k_z) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = i\mathbf{k}\cdot\mathbf{E}(\mathbf{x},t). \quad (14)$$

But in absence of charges Gauss law tells us that $\nabla \cdot \mathbf{E} = 0$ hence $\mathbf{k}\cdot\mathbf{E} = 0$. Exactly the same logic gives $\mathbf{k}\cdot\mathbf{B} = 0$ since Gilbert's law applies.

It is worth mentioning that if we take the curl of these plane wave solutions we end up with one might expect, but probably is not sure of. A short calculation reveals that $\nabla \times \mathbf{E} = i\mathbf{k}\times\mathbf{E}$ and $\nabla\times\mathbf{B} = i\mathbf{k}\times\mathbf{B}$.

Extra problem no. 4

First of all we take the curl of both sides of the Maxwell-Ampere equation and use a well known[1] identity for the curl of a curl:

$$\nabla\times\nabla\times\mathbf{B} = \mu_0\nabla\times\mathbf{J} + \epsilon_0\mu_0\frac{\partial}{\partial t}\nabla\times\mathbf{E}$$
$$\nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} = \mu_0\nabla\times\mathbf{J} + \epsilon_0\mu_0\frac{\partial}{\partial t}\nabla\times\mathbf{E}. \quad (15)$$

Now we rearrange terms and use Gilbert's and Faraday's laws to find

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = -\mu_0\nabla\times\mathbf{J}, \quad (16)$$

where $\epsilon_0\mu_0 = \frac{1}{c^2}$ also is used. Thus we have arrived at a (inhomogenous) wave equation for the magnetic field.

In order to find a wave equation for the electric field we take the curl of Faraday's law and use the same vector identity as above

$$\nabla\times\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B}$$
$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B}. \quad (17)$$

Gauss law and the Maxwell-Ampere law now yields

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0}(\nabla\rho + \frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}). \quad (18)$$

[1] If you do not know it I urge you to show if by use of index-notation.

It now remains to show that the continuity equation holds. By taking the divergence of the Maxwell-Ampere law and then applying Gauss law we immediately see that

$$\begin{aligned} 0 &= \nabla \cdot (\nabla \times \mathbf{B}) \\ &= \mu_0 \nabla \cdot \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} \\ &= \mu_0 (\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}), \end{aligned} \tag{19}$$

which shows that the electric charge is a conserved quantity. Another elegant way to prove this is to note that

$$0 = \partial_\alpha \partial_\beta F^{\alpha\beta} = 4\pi \partial_\alpha J^\alpha, \tag{20}$$

where the antisymmetric property of $F^{\alpha\beta}$ is used.

Extra problem no. 5

Suppose that we have a solution $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$. We can then use a scalar function[2] $\Lambda(x)$ that gives a new vector potential $A_\alpha(x) \to A'_\alpha(x) = A_\alpha(x) + \partial_\alpha \Lambda(x)$ since this leaves $F_{\alpha\beta}$ (and hence the physics) unchanged. This is easily proven:

$$\begin{aligned} F'_{\alpha\beta} = \partial_\alpha A'_\beta - \partial_\beta A'_\alpha &= \partial_\alpha (A_\beta + \partial_\beta \Lambda) - \partial_\beta (A_\alpha + \partial_\alpha \Lambda) \\ &= \partial_\alpha A_\beta - \partial_\beta A_\alpha + (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha)\Lambda = F_{\alpha\beta}. \end{aligned} \tag{21}$$

In order to explain the difference in arbitrariness of the gauge transformation between the Lorenz and the Coulomb gauge I will also review their basic structure.

Coulomb gauge

Given a solution A_α that does not obey the Coulomb gauge (i.e. $\partial_i A_i \neq 0$), we can find a solution A'_α that does by applying a suitable gauge transformation. We choose $\Lambda(x)$ so that $\partial_i A'_i = \partial_i A_i + \partial_i \partial_i \Lambda = 0$. We will find Λ iff we can solve

$$\nabla^2 \Lambda = -\partial_i A_i. \tag{22}$$

This is doable under reasonable conditions and we have thereby found the desired transformation. To show that the transformation is unique we assume that we had $\partial_i A_i = 0$ to begin with. To transform this solution into another solution, A'_α, that obeys the same condition we have to demand $\partial_i A'_i = \partial_i A_i + \partial_i \partial_i \Lambda = \nabla^2 \Lambda = 0$. But the uniqueness theorem for the Laplace equation tells us that the only solution is $\Lambda = 0$, thus the gauge fixing is unique.

Lorenz gauge

Given a solution A_α that does not obey the Lorenz gauge (i.e. $\partial^\alpha A_\alpha \neq 0$), we can find a solution A'_α that does by applying a suitable gauge transformation.

[2] We assume that this function is well behaved, i.e. it vanishes at infinity and is differentiable everywhere.

We choose $\Lambda(x)$ so that $\partial^\alpha A'_\alpha = \partial^\alpha A_\alpha + \partial^\alpha\partial_\alpha\Lambda = 0$. We will find Λ iff we can solve[3]

$$(\frac{\partial^2}{\partial t^2} - \nabla^2)\Lambda = -\partial^\alpha A_\alpha. \tag{23}$$

This is doable under reasonable conditions and we have thereby found the desired transformation. However, the choice of Λ is not unique in this case. To see this we just assume that we had $\partial^\alpha A_\alpha = 0$ to begin with. We can the transform to a new solution, $A_\alpha(x) \to A'_\alpha(x) = A_\alpha(x) + \partial_\alpha\Lambda(x)$, that also obeys the Lorenz gauge as long as $(\frac{\partial^2}{\partial t^2} - \nabla^2)\Lambda = 0$. This condition does not imply that $\Lambda = 0$ and therefore the gauge transformation is not unique.

Extra problem no. 6

We use the hint and vary the coordinates in the action

$$\mathcal{S} = \int dt\{-m\sqrt{1-\dot{x}_i\dot{x}_i} + e\dot{x}_iA_i + eA_0\}. \tag{24}$$

By use of the chain rule for derivatives we find that

$$\begin{aligned}\delta\mathcal{S} = \frac{\partial\mathcal{S}}{\partial x_j}\delta x_j + \frac{\partial\mathcal{S}}{\partial\dot{x}_j}\delta\dot{x}_j = \int dt\{&(e\dot{x}_i\partial_jA_i + e\partial_jA_0)\delta x_j \\ &+(\frac{m\dot{x}_j}{\sqrt{1-\dot{x}_i\dot{x}_i}} + eA_j)\delta\dot{x}_j\}.\end{aligned} \tag{25}$$

A partial integration of the second term[4] and the fact that $\frac{d}{dt}A_j = \partial_0A_j + \dot{x}_i\partial_iA_j$ now yields

$$\begin{aligned}\delta\mathcal{S} &= \int dt\{(e\dot{x}_i\partial_jA_i + e\partial_jA_0 - \frac{d}{dt}(\frac{m\dot{x}_j}{\sqrt{1-\dot{x}_i\dot{x}_i}}) - e\frac{d}{dt}A_j)\delta x_j\} \\ &= \int dt\{(e(\dot{x}_i\partial_jA_i - \dot{x}_i\partial_iA_j) + e(\partial_jA_0 - \partial_0A_j) - \frac{d}{dt}(\frac{m\dot{x}_j}{\sqrt{1-\dot{x}_i\dot{x}_i}}))\delta x_j\} \\ &= \int dt\{(e\varepsilon_{jik}\dot{x}_iB_k + eE_j - \frac{d}{dt}(\frac{m\dot{x}_j}{\sqrt{1-\dot{x}_i\dot{x}_i}}))\delta x_j\}.\end{aligned} \tag{26}$$

Since this has to hold for arbitrary variations we arrive at the equation for the Lorentz force:

$$\frac{d}{dt}(m\gamma\dot{x}_i) = e(E_i + \varepsilon_{ijk}\dot{x}_jB_k) \tag{27}$$

where we have also used the definition $\gamma \equiv \frac{1}{\sqrt{1-\dot{x}_i\dot{x}_i}}$.

It is of course also possible to derive the Lorentz force without choosing $\tau = t$. It is left as an exercise to the reader to show that the general condition is

$$\frac{d}{d\tau}(\frac{m\dot{x}_\alpha}{\sqrt{-\dot{x}_\alpha\dot{x}^\alpha}}) = e\dot{x}^\beta F_{\alpha\beta}. \tag{28}$$

[3] I use the same metric as Jackson, chapter 11.

[4] We drop the end terms since the variation is zero (by definition) at times t_1 and t_2 which we integrate between.

Extra problem no. 7

In the first part of this exercise we shall vary the field A^μ—not the coordinates x_i! We shall hence use functional derivatives $\frac{\delta}{\delta A^\mu(x)}$ which in four dimensions obey the following axiom: $\frac{\delta}{\delta A^\mu(x)} A^\mu(y) = \delta^{(4)}(x-y)$. By use of the chain rule (the generalization to a continuous set of indices) we obtain

$$\begin{aligned}
\mathcal{S}[A] &= \int d^4x(-\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} + A_\alpha J^\alpha) \\
\Rightarrow \delta\mathcal{S} &= \int d^4x(-\frac{1}{16\pi}\delta(F_{\alpha\beta}F^{\alpha\beta}) + \delta(A_\alpha)J^\alpha) \\
&= \int d^4x(-\frac{1}{8\pi}F^{\alpha\beta}\delta(F_{\alpha\beta}) + \delta(A_\alpha)J^\alpha) \\
&\{F^{\alpha\beta}(\partial_\alpha\delta(A_\beta) - \partial_\beta\delta(A_\alpha)) = -2F^{\alpha\beta}\partial_\beta\delta(A_\alpha)\} \\
&= \int d^4x(\frac{1}{4\pi}F^{\alpha\beta}\partial_\beta\delta(A_\alpha) + \delta(A_\alpha)J^\alpha) \\
&= \int d^4x\delta(A_\alpha)(-\frac{1}{4\pi}\partial_\beta F^{\alpha\beta} + J^\alpha),
\end{aligned} \tag{29}$$

where the last equality follows from a partial integration of the first term. The principle of least (or most) action tells us that $\delta\mathcal{S} = 0$ must hold for arbitrary variations in order to obtain a physical solution. It is now easy to see that this implies

$$\partial_\beta F^{\alpha\beta} = 4\pi J^\alpha, \tag{30}$$

which is the first half of Maxwell's equations. The second half follows directly from the definition $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$. To see this note that

$$\begin{aligned}
\partial_{[\alpha}F_{\beta\gamma]} &= 2(\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}) \\
= 2((\partial_\beta\partial_\gamma - \partial_\gamma\partial_\beta)A_\alpha &+ (\partial_\gamma\partial_\alpha - \partial_\alpha\partial_\gamma)A_\beta + (\partial_\alpha\partial_\beta - \partial_\beta\partial_\alpha)A_\gamma) = 0.
\end{aligned}$$

We now turn our attention to the gauge transformation $A_\alpha(x) \to A'_\alpha(x) = A_\alpha(x) + \partial_\alpha\Lambda(x)$. We plug this expression into the last line of (29) and perform a partial integration:

$$\begin{aligned}
\delta\mathcal{S} &= \int d^4x\partial_\alpha\Lambda(x)(-\frac{1}{4\pi}\partial_\beta F^{\alpha\beta} + J^\alpha) \\
&= \int d^4x\Lambda(x)(-\frac{1}{4\pi}\partial_\alpha\partial_\beta F^{\alpha\beta} + \partial_\alpha J^\alpha) \\
&= \int d^4x\Lambda(x)\partial_\alpha J^\alpha.
\end{aligned} \tag{31}$$

The above vanishes for arbitrary $\Lambda(x)$ if (and only if)

$$\partial_\alpha J^\alpha = 0, \tag{32}$$

which corresponds to the conservation of electrical charge (e.g. the continuity equation). As we have seen earlier in the course one can also deduce this result directly from Maxwell's equations.

Extra problem no. 8

Due to the lack of time I will not do this example this year. However you can try to do it yourself and you can show the expression for both length contraction and time dilation by use of extremely simple mathematics, i.e. Pythagoras theorem.

Extra problem no. 9

a) Since we want to obtain the electric and magnetic fields $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ it is sufficient to study how $F^{\alpha\beta}$ transforms. Since it is a Lorentz contravariant tensor it transforms as

$$\widetilde{F}^{\alpha\beta}(\widetilde{\mathbf{x}}) = \Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}F^{\gamma\delta}(\mathbf{x}) \quad \Leftrightarrow \quad \widetilde{\mathbf{F}}(\widetilde{\mathbf{x}}) = \mathbf{\Lambda}\mathbf{F}(\mathbf{x})\mathbf{\Lambda}^{\mathbf{T}}. \tag{33}$$

We can now write down the transformed electromagnetic field tensor in explicit matrix form:

$$\widetilde{\mathbf{F}}(\widetilde{\mathbf{x}}) = \mathbf{\Lambda}\mathbf{F}(\mathbf{x})\mathbf{\Lambda}^{\mathbf{T}} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & E_x & \gamma E_y - \gamma\beta B_z & \gamma E_z + \gamma\beta B_y \\ -E_x & 0 & \gamma B_z - \gamma\beta E_y & -\gamma B_y - \gamma\beta E_z \\ -\gamma E_y + \gamma\beta B_z & -\gamma B_z + \gamma\beta E_y & 0 & B_x \\ -\gamma E_z - \gamma\beta B_y & \gamma B_y + \gamma\beta E_z & -B_x & 0 \end{pmatrix}. \tag{34}$$

Since

$$\widetilde{\mathbf{F}} = \begin{pmatrix} 0 & \widetilde{E}_x & \widetilde{E}_y & \widetilde{E}_z \\ -\widetilde{E}_x & 0 & \widetilde{B}_z & -\widetilde{B}_y \\ -\widetilde{E}_y & -\widetilde{B}_z & 0 & \widetilde{B}_x \\ -\widetilde{E}_z & \widetilde{B}_y & -\widetilde{B}_x & 0 \end{pmatrix} \tag{35}$$

we can now identify

$$\widetilde{\mathbf{E}} = \begin{pmatrix} E_x \\ \gamma E_y - \gamma\beta B_z \\ \gamma E_z + \gamma\beta B_y \end{pmatrix}; \quad \widetilde{\mathbf{B}} = \begin{pmatrix} B_x \\ \gamma B_y + \gamma\beta E_z \\ \gamma B_z - \gamma\beta E_y \end{pmatrix}. \tag{36}$$

It is essential to note that the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are still expressed in terms of coordinates x_i, i.e. not in the coordinate system of $\widetilde{S}$!

b) In frame S we have $E_i = \frac{x_i}{4\pi\epsilon_0|x|^3}$ and $B_i = 0$. According to what we found in the first part of this exercise we can now write down the answer in terms of the old coordinates:

$$\widetilde{\mathbf{E}} = \frac{1}{4\pi\epsilon_0|x|^3}\begin{pmatrix} x \\ \gamma y \\ \gamma z \end{pmatrix}; \quad \widetilde{\mathbf{B}} = \frac{1}{4\pi\epsilon_0|x|^3}\begin{pmatrix} 0 \\ \gamma\beta z \\ -\gamma\beta y \end{pmatrix}. \tag{37}$$

It now remains to change to the coordinates $\widetilde{x}_i$ that an observer in $\widetilde{S}$ would use. We find that

$$\mathbf{x} = \mathbf{\Lambda^{-1}}\widetilde{\mathbf{x}} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{t} \\ \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} = \begin{pmatrix} \gamma t + \gamma\beta x \\ \gamma x + \gamma\beta t \\ y \\ z \end{pmatrix} \tag{38}$$

and insert this result into (37) to obtain

$$\begin{aligned} \widetilde{\mathbf{E}} &= \frac{\gamma}{4\pi\epsilon_0(\gamma^2(\widetilde{x}+\beta\widetilde{t})^2+\widetilde{y}^2+\widetilde{z}^2)^{3/2}} \begin{pmatrix} \widetilde{x}+\beta\widetilde{t} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix}, \\ \widetilde{\mathbf{B}} &= \frac{\gamma\beta}{4\pi\epsilon_0(\gamma^2(\widetilde{x}+\beta\widetilde{t})^2+\widetilde{y}^2+\widetilde{z}^2)^{3/2}} \begin{pmatrix} 0 \\ \widetilde{z} \\ -\widetilde{y} \end{pmatrix}, \end{aligned} \tag{39}$$

which is what we were looking for.

Extra problem no. 10

By definition a scalar field transforms according to

$$\widetilde{\phi}(\widetilde{\mathbf{x}}) = \phi(\mathbf{x}) \quad \Leftrightarrow \quad \widetilde{\phi}(\mathbf{x}) = \phi(\mathbf{\Lambda^{-1}}\mathbf{x}). \tag{40}$$

We now make use of (38) and can immediately write down the answer since we know that $\phi(\mathbf{x}) = \frac{1}{r^2}$:

$$\widetilde{\phi}(\mathbf{x}) = \frac{1}{(\gamma^2(x+\beta t)^2+y^2+z^2)^{3/2}}. \tag{41}$$