Final Examination Paper for Electrodynamics-I

Date: *Friday, Nov 02, 2012,* Time: *09:00 - 14:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks or equivalent*

Questions:	1	2	3	4	5	Total
Marks:	16	16	16	16	16	80

Please explain your reasoning and calculations clearly

1. (a) Consider a charge q placed a distance $\vec{y}$ from a charge Q. Express the potential $\Phi(\vec{x})$ in terms multipole moments of the distribution, retaining the first two moments (you can choose the midpoint of the charge system as the origin of the coordinate system).

   (b) For a charge distribution ρ, define the multipole moments q_{lm} in spherical polar coordinates. (i) Show that for a spherically symmetric charge distribution, all multipole moments beyond the monopole moment vanish. (ii) Show that for a charge distribution with axial symmetry all moments except q_{l0} vanish.

Solution (points: 8+8)

a) The potential for the system (choosing the mid point as the origin of the coordinate system) is given by

$$\Phi(\vec{x}) = \frac{q}{|\vec{x} - \vec{y}/2|} + \frac{Q}{|\vec{x} + \vec{y}/2|} .$$

Expanding in powers of $1/x$ (where $x = |\vec{x}|$),

$$\frac{1}{|\vec{x} \mp \vec{y}/2|} = \frac{1}{x} \pm \frac{\vec{y} \cdot \vec{x}}{2x^3} + \cdots .$$

Then,

$$\Phi(\vec{x}) = \frac{q + Q}{x} + \frac{(q - Q)\vec{y} \cdot \vec{x}}{2x^3} + \cdots .$$

This system has a monopole moment $q + Q$ and a dipole moment $\vec{p} = (q - Q)\vec{y}/2$.

b) The multipole moments q_{lm} of the charge distribution $\rho(\vec{x})$ are given by

$$q_{lm} = \int d^3x' \rho(\vec{x}\,') r'^l Y^*_{lm}(\theta', \phi')$$

(i) For a spherically symmetric charge distribution, $\rho(\vec{x}) \equiv \rho(r, \theta, \phi) = \rho(r)$, independent of the angular variables. Therefore the integral factorizes into radial and angular parts,

$$q_{lm} = \left(\int_0^\infty r'^2 dr' \rho(r') r'^l \right) \left(\int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' Y^*_{lm}(\theta', \phi') \right)$$

Since $Y_{00}(\theta', \phi') = 1/\sqrt{4\pi}$, we can insert $Y_{00}(\theta', \phi')\sqrt{4\pi} = 1$ in the angular integration. Now, from the orthogonality property of spherical harmonics it follows that the angular integral is proportional to δ_{l0} and hence vanishes for all $l \geq 1$ ($l = 0$ being the monopole moment).

(ii) In the case of axial symmetry about the z-axis, ρ is independent of the azimuthal coordinate ϕ. In this case,

$$q_{lm} = (const)\left(\int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \rho(r', \theta') r'^l P_l^m(\cos\theta')\right)\left(\int_0^{2\pi} d\phi' e^{-im\phi'}\right)$$

where we have used the fact that $Y_{lm}(\theta, \phi) = (const) P_l^m(\cos\theta) e^{im\phi}$. Now, the ϕ integral gives a δ_{m0}. Thus the only non-vanishing moments in this case are q_{l0}.

2. (a) As charges move into or out of a volume V, the total charge Q_V contained in it changes. Show that the assumption of charge conservation leads to the continuity equation $\partial_\mu J^\mu = 0$.

   (b) Consider a current distribution with local current density $J(\vec{x})$ in a volume V placed in an external magnetic field $\vec{B}(\vec{x})$. (i) Find the expression for the net force $\vec{F}$ acting on the current distribution. (ii) If this current flows in a long, straight, thin wire placed at angle θ with respect to a uniform magnetic field, compute the force per unit length on the wire. (iii) If this force moves the wire, compute the work done by the magnetic field.

   (c) Describe magnetic hysteresis and its origin.

Solution (points: 5+6+5)

a) The rate of change of charge in volume V is $\partial Q_V/\partial t$. Since total charge is conserved, this change can take place only if some charge leaves or enters volume V. This give rise to a current I_S across the boundary of V and by charge conservation,

$$\frac{\partial Q_V}{\partial t} = -I_S$$

The negative sign corresponds to the convention that positive charge leaving volume V gives rise to a positive current across its boundary. Let's express Q_V and I_S in terms of charge and current densities,

$$Q_V = \int_V d^3x \rho\,, \qquad I_S = \oint\!\!\!\oint_S \vec{J} \cdot d\vec{S} = \int_V d^3x \vec{\nabla} \cdot \vec{J}$$

where the divergence theorem is used in the last step. This gives

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0\,.$$

Denoting $x^0 = ct$ and $J^0 = c\rho$, this becomes, $\partial_\mu J^\mu = 0$.

b) (i) The Lorentz force acting on a volume element d^3x within the current distribution is $d\vec{F} = \frac{1}{c}(\rho d^3x)\vec{v} \times \vec{B}$. Then the total force on the volume V is

$$\vec{F} = \frac{1}{c}\int_V \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x\,.$$

(ii) For a wire we can write $d^3x = \vec{ds}\cdot\vec{dl}$ with $\vec{dl}$ along the length of the wire and $\vec{ds}$ a surface element over the cross section of the wire. In the thin wire approximation, $\vec{J}$ is parallel to $\vec{dl}$ and the variation of $\vec{B}$ over the cross section can be neglected. Then, $\int_V d^3x\vec{J}(\vec{x})\times\vec{B}(\vec{x}) = \int_l\int_S(\vec{ds}\cdot\vec{dl})(\vec{J}\times\vec{B}) = \int_l\int_S(\vec{ds}\cdot\vec{J})(\vec{dl}\times\vec{B}) = I\int_l\vec{dl}\times\vec{B}$. Then, for a long straight wire of length L, say in the z-direction, and a uniform field $\vec{B}$ that does not vary along the length of the wire, one obtains the force per unit length as

$$\frac{\vec{F}}{L} = \frac{I}{c}\,\hat{z}\times\vec{B}$$

(iii) The force is perpendicular to $\hat{z}$ and hence moves the wire parallel to itself in the $x-y$ plane. For a displacement $\vec{dr}$ in this plane, the work done is $\vec{F}\cdot\vec{dr}\neq 0$. While a magnetic field does no work on free moving charges, that end up moving in helical paths, the work is non-zero when the charges are confined to move in a straight wire.

c) In a magnetic medium atoms and molecules could carry magnetic dipole moments. In an external magnetic field, these microscopic magnetic dipoles tend to realign themselves giving rise to a net magnetic field that is characterized by the magnetic moment density or magnetization $\vec{M}$ of the medium. The total magnetic field $\vec{B}$ is a sum of the field produced by $\vec{M}$ and the externally applied field $\vec{H}$. These are related by $\vec{H} = \vec{B} - 4\pi\vec{M}$ and it can be shown to satisfy $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}$ where $\vec{J}$ is the external "free" current density. Magnetic hysteresis describes the variation of the magnetic field in ferromagnets as a function of a varying applied current.

Since H is determined by the applied current, one can alternatively consider B as a function of H. For diamagnetic and paramagnetic material the relation between the two is linear to a good degree of approximation, $B = \mu H$. However, ferromagnetic material respond strongly to the applied field and the relation $B = B(H)$ has the following features: Initially, as H is increased from zero, B also increases due to the fact that more and more microscopic magnetic dipoles in the material realign themselves along the applied field, enhancing its strength. But when most of the microscopic dipoles have realigned, the increase in B slows down and the B vs H curve flattens out. Now, as H is decreased, B also decreases, but it starts lagging behind H since the microscopic dipoles resist flipping their orientation. Even when H is reduced to zero, there is still a residual B field. As H is made negative, B reduces further and finally vanishes for some negative H. Beyond this, B increases in the negative direction until most of the microscopic dipoles are aligned in the new direction and beyond some negative H the curve again flattens out. In this way, as H is varied sinusoidally, B traces a closed loop in the $B-H$ plane called the Hysteresis curve (it can be found in any standard textbook on electromagnetism) and the phenomenon is referred to as hysteresis.

3. (a) Starting with the homogeneous Maxwell's equations find the expression for $\vec{E}$ and $\vec{B}$ in terms of the potentials $\vec{A}$ and ϕ.

(b) Obtain the equations that the potentials satisfy in the Lorenz gauge ($\nabla\times(\nabla\times\vec{Q}) = -\nabla^2\vec{Q} + \nabla(\nabla\cdot\vec{Q})$).

(c) Write the plane wave solutions for the potentials and show that the $\vec{E}$ and $\vec{B}$ obtained from them are transverse to each other and to the direction of propagation.

Solution (points: 5+5+6)
a) The homogeneous Maxwell's equations are $\vec{\nabla}\cdot\vec{B} = 0$ *and* $\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}\vec{B} = 0$. *The first equation implies* $\vec{B} = \vec{\nabla}\times\vec{A}$, *for some vector field* $\vec{A}$. *With this, the second equation becomes* $\vec{\nabla}\times(\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}\vec{A}) = 0$ *which implies* $(\vec{E}+\frac{1}{c}\frac{\partial}{\partial t}\vec{A}) = -\vec{\nabla}\Phi$, *for some scalar potential* Φ. *Then we have* $\vec{E}$ *and* $\vec{B}$ *in terms of the potentials* $\vec{A}$ *and* Φ *as,*

$$\vec{B} = \vec{\nabla}\times\vec{A}\,, \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$

b) The two inhomogeneous Maxwell equations are, $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ *and* $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{J}$. *Substituting for* $\vec{E}$ *and* $\vec{B}$ *in terms of the potentials leads to the coupled equations,*

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho\,,$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\Phi) = -\frac{4\pi}{c}\vec{J}.$$

In the Lorenz gauge, $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\Phi = 0$, *the above equations reduce to,*

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\Phi = -4\pi\rho\,, \qquad (\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\vec{A} = -\frac{4\pi}{c}\vec{J}.$$

c) The above equations have plane-wave solutions in the absence of sources,

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\Phi = 0\,, \qquad (\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\vec{A} = 0$$

The monochromatic plane-wave solutions in the direction $\vec{k}$ *are,*

$$\Phi = \Phi_0\exp^{i(\vec{k}\cdot\vec{x}-\omega t)}\,, \qquad \vec{A} = \vec{A}_0\exp^{i(\vec{k}\cdot\vec{x}-\omega t)}\,,$$

where,

$$\omega = \pm\, c\,|\vec{k}|\,.$$

The Lorenz gauge condition implies $\vec{k}\cdot\vec{A}_0 = (\omega/c)\Phi_0$. *The magnetic and electric fields become,* $\vec{B} = i\vec{k}\times\vec{A}$ *and* $\vec{E} = -i\vec{k}\Phi + i\frac{\omega}{c}\vec{A}$ *(the physical fields are the real parts of these). Note that* $\vec{B}$ *is normal to both* $\vec{k}$ *and* $\vec{A}$, *hence it is normal to* $\vec{E}$ *and we have* $\vec{B}\cdot\vec{E} = 0$. *This also implies* $\vec{k}\cdot\vec{B} = 0$. *On the other hand,* $\vec{k}\cdot\vec{E} = -i(k^2\Phi - \frac{\omega}{c}\vec{k}\cdot\vec{A}) = -i(k^2 - \frac{\omega^2}{c^2})\Phi = 0$.

4. Obtain the expression for the Poynting theorem. Clarify the physical meaning of the various terms in the equation and show that the Poynting vector represents energy flux carried by electric and magnetic fields.

Solution (points: 16)

To obtain the Poynting theorem, we consider a region of space with current distribution $\vec{J}$ in the presence of electric and magnetic fields $\vec{E}$ and $\vec{B}$. Start with $\vec{J} \cdot \vec{E}$ and, using Maxwell's equations, rewrite $\vec{J}$ in terms of $\vec{E}$ and $\vec{B}$. After some manipulations (that should be carried out explicitly), one gets,

$$\vec{J} \cdot \vec{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon \vec{E} \cdot \vec{E} + \frac{1}{\mu}\vec{B} \cdot \vec{B}\right) + \frac{c}{4\pi}\nabla \cdot (\vec{E} \times \vec{H}) = 0$$

In the above we recognize the following terms: $\vec{J} \cdot \vec{E}$ is the power per unit volume injected into the current distribution by the electric field. $u = \frac{1}{8\pi}\left(\epsilon \vec{E} \cdot \vec{E} + \frac{1}{\mu}\vec{B} \cdot \vec{B}\right)$ is the energy density contained in the electric and magnetic fields. The equation states that this energy density u changes partly because some energy is transferred to the current system through $\vec{J} \cdot \vec{E}$ and partly because of some energy transfer described by the quantity $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{H}$. To see clearly what this means, integrate the expression over a volume V of boundary ∂V. Then using the divergence theorem, one gets,

$$\int_V d^3x \left(\vec{J} \cdot \vec{E} + \frac{\partial}{\partial t}(u)\right) + \int_{\partial V} \vec{da} \cdot \vec{S} = 0$$

Then clearly the Poynting vector $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{H}$ is an energy flux leaving or entering the volume V through its boundary ∂V to maintain energy conservation within V. This is the energy transported by the electric and magnetic fields.

5. (a) Define contravariant and covariant vectors with respect to general coordinate transformations.

   (b) For a particle of rest mass m_0 moving with velocity $\vec{u}$ in an inertial frame one can define a relativistic 4-velocity $U^\mu = (\gamma c, \gamma \vec{u})$, where $\gamma^{-1} = \sqrt{1 - u^2/c^2}$. Now, for a point charge q consider the equation,

   $$m_0 \frac{dU^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} U_\nu \,,$$

   where, τ is the time in the rest-frame of the moving charge ($dt = \gamma d\tau$). Write down the spatial and temporal components of this equation in terms of electric and magnetic fields and describe the physical meaning of the resulting equations.

   (c) Find the electric and magnetic potentials, $\Phi, \vec{A}$, produced by a charged particle in uniform motion, moving with speed u in the x^1 direction.

Solution (points: 4+6+6)

a) Under a general coordinate transformation $x^\mu \to \tilde{x}^\mu$, in general, covariant vectors W_μ and contravariant vectors V^μ transform in the same way as ∂_μ and dx^μ.

$$\widetilde{W}_\mu(\widetilde{x}) = \frac{\partial x^\nu}{\partial \widetilde{x}^\mu} W_\nu(x) \,, \quad \widetilde{V}^\mu(\widetilde{x}) = \frac{\partial \widetilde{x}^\mu}{\partial x^\nu} V^\nu(x)$$

b) Spatial components: For $\mu = i$, the relativistic equation reduces to

$$m\frac{du^i}{dt} + \frac{m}{\gamma}\frac{d\gamma}{dt}u^i = qF^{i0} - \frac{q}{c}F^{ij}\,u_j$$

But, $F^{i0} = E^i$, $F^{ij} = -\epsilon^{ij}_{\ \ k}B^k$ and $(\vec{u}\times\vec{B})^i = -\epsilon^i_{\ jk}u^jB^k$ (with $\epsilon_{123} = 1$), and therefore,

$$m\frac{d\vec{u}}{dt} + \frac{m}{\gamma}\frac{d\gamma}{dt}\vec{u} = q\vec{E} + \frac{q}{c}\vec{u}\times\vec{B}$$

where $m = \gamma m_0$. In terms of the momentum $\vec{p} = m_0\gamma\vec{u}$, it becomes,

$$\frac{d\vec{p}}{dt} = q\vec{E} + \frac{q}{c}\vec{u}\times\vec{B}$$

Note that γ being a function of the velocity $\vec{u}$ of the moving particle, is not constant in time.
Temporal component: For $\mu = 0$, it reduces to $d(m_0c^2\gamma)/dt = qE^iu^i$. We recognize $\mathcal{E} = m_0c^2\gamma$ as the relativistic energy of the particle. Hence $d(\mathcal{E})/dt = q\vec{u}\cdot\vec{E}$ which gives the power transferred to the charged particle from the electric field.
c) The electric potential $\phi(x)$ and magnetic potential $\vec{A}(x)$ combine into a 4-vector $A^\mu = \{A^0 = \phi, \vec{A}\}$ which under Lorentz transformations L transforms as

$$\tilde{A}^\mu(\tilde{x}) = L^\mu_{\ \nu}A^\nu(L^{-1}\tilde{x})$$

In our case, ϕ and $\vec{A} = 0$ are the fields in the rest frame of the particle. The non-trivial components of L are, $L^0_{\ 0} = L^1_{\ 1} = \gamma$, and $L^1_{\ 0} = L^0_{\ 1} = -\gamma\beta$. Therefore Lorentz transformation gives (suppressing the $\tilde{x}$ dependence)

$$\tilde{\phi} = \gamma\phi\,, \qquad \tilde{A}^1 = -\gamma\beta\phi\,, \qquad \tilde{A}^2 = \tilde{A}^3 = 0$$

To complete the transformation, we have to express the x^μ dependence of ϕ in terms of $\tilde{x}^\mu$. For the given Lorentz transformation, $x^1 = \gamma(\tilde{x}^1 + \beta\tilde{x}^0)$, $x^2 = \tilde{x}^2$ and $x^3 = \tilde{x}^3$, so that $x^2 = \sum_1^3 x^ix^i = \gamma^2(\tilde{x}^1 + u\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2$. Then,

$$\tilde{\phi}(\tilde{x}) = \gamma\frac{Q}{\sqrt{(\gamma^2(\tilde{x}^1 + u\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2)}}\,, \qquad \tilde{A}^1 = -\beta\tilde{\phi}(\tilde{x})$$