Final Examination Paper for Electrodynamics-I

Date: *Friday, Oct 28, 2011*, Time: *09:00 - 14:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks or equivalent*

Questions:	1	2	3	4	5	Total
Marks:	16	16	16	16	16	80

Note: Please explain your reasoning and calculations clearly

1. (a) Prove Green's first and second identities.

   (b) Using Green's second identity describe how boundary conditions are incorporated in the solutions of the Poisson equation.

   Solution (points: 8+8)
   a) & b) See Jackson's book, sections 1.8 and 1.10 (or the lecture notes, set-II).

2. (a) Consider two neutral media with dielectric constants ϵ_1 and ϵ_2, in contact through a surface S. After an electric field is switched on, compute the induced surface charge density on S in terms of the value of the electric field in one of the media. Also describe the physical origin of the induced surface charge.

   (b) Consider solutions of the Laplace equation, $\nabla^2\Phi(r,\theta,\phi) = 0$, in situations with *azimuthal symmetry* (derivation not needed). Describe in what sense the r-dependence of the solution uniquely determines its θ-dependence?

   Solution (points: 8+8)
   a) Since there are no free charges on the surface S, the electric displacement vectors on the two sides of it satisfy $(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = 0$ implying $(\epsilon_2\vec{E}_2 - \epsilon_1\vec{E}_1)\cdot\hat{n} = 0$. But the induced polarization charges produce a discontinuity in the electric field given by the induced surface charge density, $(\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = 4\pi\sigma_{ind}$. Hence, $\sigma_{ind} = \frac{1}{4\pi}(\frac{\epsilon_1}{\epsilon_2}-1)\vec{E}_1\cdot\hat{n} = \frac{1}{4\pi}(1-\frac{\epsilon_2}{\epsilon_1})\vec{E}_2\cdot\hat{n}$. The origin of σ_{ind} is in the unequal polarizations of the two media. In each medium, the applied electric field causes a small displacement of the positive and negative "charge clouds" with respect to each other. Since these displacements in the two media are unequal, there is an accumulation of charge on the boundary where the behaviour changes.
   b) The general solution of the Laplace equation in spherical polar coordinates is,

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^l + B_{lm}r^{-l-1}\right)\,Y_{lm}(\theta,\phi)\,,$$

   in terms of the spherical harmonics $Y_{lm}(\theta,\phi) \sim P_l^m(\cos\theta)e^{im\phi}$. A given power of r is associated with different θ dependences depending on the value of m. In problems *with azimuthal symmetry, Φ is independent of ϕ. Hence all A_{lm} and B_{lm} for $m \neq 0$ vanish and the solution reduces to*

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\theta)\,.$$

Clearly here any power of r is associated with a unique θ dependence given by the Legendre polynomial. Hence, knowing the r dependence, the θ dependence can be uniquely determined.

3. (a) Consider a current distribution with local current density $\vec{J}(\vec{x})$ in a volume V placed in an external magnetic field $\vec{B}(\vec{x})$. Find the expression for the net force $\vec{F}$ acting on the current distribution.

   (b) Work out the expression for the force acting between two thin wires carrying currents I_1 and I_2 (the wires are not necessarily straight and parallel).

   Solution (points: 8+8)
   a) The Lorentz force acting on a volume element d^3x within the current distribution is $d\vec{F} = \dfrac{1}{c}(\rho d^3x)\vec{v}\times\vec{B}$ so that the force on the volume V is $\vec{F} = \frac{1}{c}\int_V \vec{J}(\vec{x})\times\vec{B}(\vec{x})\,d^3x$
   b) For a wire we can write $d^3x = \vec{ds}\cdot\vec{dl}$ with $\vec{dl}$ along the length of the wire and $\vec{ds}$ a surface element over the cross section of the wire. In the thin wire approximation, $\vec{J}$ is parallel to $\vec{dl}$ and the variation of $\vec{B}$ over the cross section can be neglected. Then, $\int_V d^3x\vec{J}(\vec{x})\times\vec{B}(\vec{x}) = \int_l\int_S(\vec{ds}\cdot\vec{dl})(\vec{J}\times\vec{B}) = \int_l\int_S(\vec{ds}\cdot\vec{J})(\vec{dl}\times\vec{B}) = I_1\int_{l_1}\vec{dl}\times\vec{B}$. This gives the force on wire 1 due to the field $\vec{B}$. Also in this approximation and using the same arguments, the magnetic field produced by a current I_2 becomes, $\vec{B}(\vec{x}) = \frac{I_2}{c}\int_{l_2}\vec{dl}'\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$. Using this in the force expression gives,

$$F_{12} = \frac{I_1 I_2}{c^2}\int_{l_1}\int_{l_2}\vec{dl}\times\vec{dl}'\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

4. (a) Consider the monochromatic plane-wave solutions of the source-free Maxwell equations in a medium, say, of the form $\vec{E}_0\cos(\vec{k}\cdot\vec{x}-\omega t)$ and $\vec{B}_0\cos(\vec{k}\cdot\vec{x}-\omega t)$ with $\omega/k = c/\sqrt{\epsilon\mu}$. Verify that the Poynting vector reproduces the expected energy current, $\vec{S} = \vec{v}\,u$, where u is the energy density of the electromagnetic field and $\vec{v}$ its propagation velocity.

   (b) Derive the Poynting theorem. Explain what it means physically and describe the meaning of each term in the expression.

   Solution (points: 8+8)
   a) The energy density of $\vec{E}$ and $\vec{B}$ is $u = \frac{1}{8\pi}(\epsilon E^2 + \frac{1}{\mu}B^2)$. Demanding that the given solutions satisfy the Maxwell's equations implies that $\vec{E}$, $\vec{B}$ and $\vec{k}$ are all perpendicular to each other and also that $|\vec{B}| = \sqrt{\mu\epsilon}|\vec{E}|$. Hence for these solutions, $u = \frac{\epsilon}{4\pi}E^2$. On the other hand, in this case the Poynting vector becomes $\vec{S} \equiv \frac{c}{4\pi\mu}\vec{E}\times\vec{B} = \frac{c}{4\pi\mu}\sqrt{\mu\epsilon}\,E^2\,\hat{k}$. Expressed in terms of u, $\vec{S} = \frac{c}{\sqrt{\mu\epsilon}}u\,\hat{k}$. This is the expected

result on realizing that $\frac{c}{\sqrt{\mu\epsilon}}\hat{k}$ *is the propagation velocity of light in the medium.*
b) To obtain the Poynting theorem, start with $\vec{J}\cdot\vec{E}$ *and, using Maxwell's equations, rewrite* $\vec{J}$ *in terms of* $\vec{E}$ *and* $\vec{B}$*. After some manipulations, one gets,*

$$\vec{J}\cdot\vec{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon\vec{E}\cdot\vec{E} + \frac{1}{\mu}\vec{B}\cdot\vec{B}\right) + \frac{c}{4\pi}\nabla\cdot(\vec{E}\times\vec{H}) = 0$$

The various terms have the following meanings:
$\vec{J}\cdot\vec{E}$*: power injected into the current distribution by the electric field/unit volume.*
$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon\vec{E}\cdot\vec{E} + \frac{1}{\mu}\vec{B}\cdot\vec{B}\right)$*: Rate of change of energy densities of the electric and magnetic fields.*
$\frac{c}{4\pi}\nabla\cdot(\vec{E}\times\vec{H})$*: Energy flux per unit time per unit volume carried by the electromagnetic fields. This is the Poynting vector. That it corresponds to the energy flux per unit area per unit time across a surface follows from the divergence theorem,* $\int_V d^3x\nabla\cdot(\vec{E}\times\vec{H}) = \int_S \vec{dS}\cdot(\vec{E}\times\vec{H})$.

The Poynting theorem is a statement of conservation of energy (as is obvious form the meaning of the various terms appearing in it). It also provides a mathematical expression for the energy carried by electromagnetic waves in the form of the Poynting vector.

5. (a) Consider a linear transformation $\widetilde{x}^\mu = L^\mu{}_\nu x^\nu$. Derive the condition that the invariance of spacetime intervals impose on the matrix L.

   (b) Consider a frame $\widetilde{S}$ moving away from a frame S with velocity v in the x^1 direction. If the observer in S measures an electric field $\vec{E}$ and no mangnetic field, derive the expression for the fields $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ that the observer in $\widetilde{S}$ will measure.

   (c) Find the electric and magnetic fields produced by a charged particle in uniform motion, moving in the x^1 direction.

Solution (points: 4+6+6)
a) The space time interval is $(\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 \equiv \Delta x^\mu \eta_{\mu\nu}\Delta x^\nu$*. In matrix notion, this can be written as* $(\Delta x)^T\eta\Delta x$*. Under the given transformation,* $\Delta\widetilde{x}^\mu = L^\mu{}_\nu\Delta x^\nu$*. The invariance of the interval,* $\Delta\widetilde{x}^\mu\eta_{\mu\nu}\Delta\widetilde{x}^\nu = \Delta x^\mu\eta_{\mu\nu}\Delta x^\nu$ *then implies,*

$$L^T\eta L = \eta$$

b) The Lorentz transformation of electric and magnetic fields $\vec{E}$ *and* $\vec{B}$ *are contained in* $\widetilde{F}^{\mu\nu}(\widetilde{x}) = L^\mu{}_\rho F^{\rho\sigma}(x)(L^T)_\sigma{}^\nu$*, or simply, in* $\widetilde{F}(\widetilde{x}) = L\,F(x)\,L^T$*. In this problem,*

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & 0 & 0 \\ E^2 & 0 & 0 & 0 \\ E^3 & 0 & 0 & 0 \end{pmatrix}.$$

Then carrying out the multiplication, one obtains the fields,

$$\begin{array}{rclcrcl} \widetilde{E}^1 &=& E^1 & \qquad & \widetilde{B}^1 &=& 0 \\ \widetilde{E}^2 &=& \gamma E^2 & & \widetilde{B}^2 &=& \gamma\beta E^3 \\ \widetilde{E}^3 &=& \gamma E^3 & & \widetilde{B}^3 &=& -\gamma\beta E^2 \end{array}$$

c) To solve this, we use the answer to part (b). Consider a reference frame S moving along with the charged particle (and for convenience, centered at the location of the particle). In this frame, the particle is at rest and its electric field is given by $\vec{E}(\vec{x}) = q\vec{x}/|x|^3$. This is time independent and the magnetic field vanishes. Take $\widetilde{S}$ to be the frame of the observer in which the particle (and hence the frame S) moves with velocity u in the x^1 direction. Then $\widetilde{S}$ moves with respect to S with velocity $v = -u$. The fields observed by an observer in $\widetilde{S}$ are given in the answer to part (b) above with $v = -u$. But we have to express the x^μ, on which $\vec{E}$ depends, in terms of $\widetilde{x}^\mu$. That gives, $x^1 = \gamma(\widetilde{x}^1 + \beta\widetilde{x}^0)$, where $\beta = v/c = -u/c$, and

$$|x|^3 \equiv [(x^1)^2 + (x^2)^2 + (x^3)^2]^{3/2} = [\gamma^2(\widetilde{x}^1 + \beta\widetilde{x}^0)^2 + (\widetilde{x}^2)^2 + (\widetilde{x}^3)^2]^{3/2} .$$

Explicitly,

$$\begin{array}{rclcrcl} \widetilde{E}^1 &=& \gamma q \, \dfrac{(\widetilde{x}^1 - u\widetilde{x}^0/c)}{|x|^3} \, , & \qquad & \widetilde{B}^1 &=& 0 \, , \\ \widetilde{E}^2 &=& \gamma q \, \dfrac{\widetilde{x}^2}{|x|^3} \, , & & \widetilde{B}^2 &=& -\gamma u q \, \dfrac{\widetilde{x}^3}{c|x|^3} \, , \\ \widetilde{E}^3 &=& \gamma q \, \dfrac{\widetilde{x}^3}{|x|^3} \, , & & \widetilde{B}^3 &=& \gamma u q \, \dfrac{\widetilde{x}^2}{c|x|^3} \, , \end{array}$$

where $|x|^3$ is given above.