Final Examination Paper for Electrodynamics-I

Date: *Friday, Oct 29, 2010*, Time: *09:00 - 14:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks or equivalent*

Questions:	1	2	3	4	5	6	Total
Marks:	13	13	13	13	14	14	80

Note: Please explain your reasoning and calculations clearly

1. (a) Show that the electrostatic equation $\vec{\nabla} \times \vec{E} = 0$ implies that the electric field $\vec{E}$ can always be obtained from a potential as $\vec{E} = -\vec{\nabla}\Phi$. Show that $\Phi(\vec{r})$ is the potential energy per unit charge.

   (b) Consider solutions of the Laplace equation, $\nabla^2\Phi(r,\theta,\phi) = 0$, in situations with *azimuthal symmetry* (derivation not needed). Describe in what sense the r-dependence of the solution uniquely determines its θ-dependence?

   Solution (points: 8+5)

   a) Stokes's theorem leads to $\oint_l \vec{E} \cdot \vec{dl} = 0$ for a closed path l, which shows the path independence of the integral. To see this, split the path l into two parts, $l = l_{AB} + l_{BA}$. Keeping l_{BA} fixed and varying l_{AB} one sees that $\int_{r_A}^{r_B} \vec{E} \cdot dl (= -\int_{r_B}^{r_A} \vec{E} \cdot dl)$ remains unchanged which implies that it can only depend on the end points of the path $(\vec{r}_A, \vec{r}_B)$ and not on the actual shape of the path connecting these two points, that is, $\int_{r_A}^{r_B} \vec{E} \cdot \vec{dl} = \Phi(\vec{r}_A, \vec{r}_B)$. A small variation of the point B, $\vec{r}_B \to \vec{r}_B + \delta\vec{r}_B$ results in $\int_{r_A}^{r_B+\delta r_B} \vec{E} \cdot \vec{dl} = \Phi(\vec{r}_A, \vec{r}_B + \delta r_B)$. Comparing the two equations one sees that for a small δr_B, the variation of the left hand side is $\vec{E} \cdot \delta\vec{r}_B$ while that of the right hand side is $\vec{\nabla}\Phi \cdot \delta\vec{r}_B$. Since this is true for any small $\delta\vec{r}_B$, one concludes that $\vec{E} = -\nabla\phi$ (the extra negative sign introduced here is a matter of convention and does not change the result). Now, work done in moving a charge q between points 1 and 2 against an electric field is $W_{21} = -q\int_1^2 \vec{E} \cdot \vec{dl} = q\int_1^2 \vec{\nabla}\Phi \cdot \vec{dl} = q(\Phi(\vec{x}_2) - \Phi(\vec{x}_1))$. Hence $\Phi(\vec{x})$ has the interpretation of electric potential energy per unit charge.

   b) The general solution of the Laplace equation in spherical polar coordinates is given by

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + B_{lm} r^{-l-1}\right) Y_{lm}(\theta,\phi)\,,$$

   in terms of the spherical harmonics $Y_{lm}(\theta,\phi) \sim P_l^m(\cos\theta)e^{im\phi}$. A given power of r is associated with different θ dependences depending on the value of m. In problems with azimuthal symmetry, Φ is independent of ϕ. Hence all A_{lm} and B_{lm} for $m \neq 0$ vanish and the solution reduces to

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\theta)\,.$$

Clearly here any power of r is associated with a unique θ dependence given by the Legendre polynomial. Hence, knowing the r dependence, the θ dependence can be uniquely determined.

2. (a) Consider two neutral media with dielectric constants ϵ_1 and ϵ_2, in contact through a surface S. After an electric field is switched on, compute the induced surface charge density on S in terms of the value of the electric field in one of the media. Also describe the physical origin of the induced surface charge.

   (b) Consider a charge q placed a distance $\vec{y}$ from a charge Q. Express the potential $\Phi(\vec{x})$ in terms multipole moments of the distribution, retaining the first two moments (you can choose the midpoint of the charge system as the origin of the coordinate system).

Solution (points: 6+7)

a) Since there are no free charges on the surface S, the electric displacement vectors on the two sides of it satisfy $(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = 0$ implying $(\epsilon_2\vec{E}_2 - \epsilon_1\vec{E}_1)\cdot\hat{n} = 0$. But the induced polarization charges produce a discontinuity in the electric field given by the induced surface charge density, $(\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = 4\pi\sigma_{ind}$. Hence, $\sigma_{ind} = \frac{1}{4\pi}(\frac{\epsilon_1}{\epsilon_2}-1)\vec{E}_1\cdot\hat{n} = \frac{1}{4\pi}(1-\frac{\epsilon_2}{\epsilon_1})\vec{E}_2\cdot\hat{n}$. The origin of σ_{ind} is in the unequal polarizations of the two media. In each medium, the applied electric field causes a small displacement of the positive and negative "charge clouds" with respect to each other. Since these displacements in the two media are unequal, there is an accumulation of charge on the boundary where the behaviour changes.

b) The potential for the system (choosing the mid point as the origin of the coordinate system) is given by

$$\Phi(\vec{x}) = \frac{q}{|\vec{x}-\vec{y}/2|} + \frac{Q}{|\vec{x}+\vec{y}/2|}\,.$$

Expanding in powers of $1/x$ (where $x = |\vec{x}|$),

$$\frac{1}{|\vec{x}\mp\vec{y}/2|} = \frac{1}{x} \pm \frac{\vec{y}\cdot\vec{x}}{2x^3} + \cdots\,.$$

Then,

$$\Phi(\vec{x}) = \frac{q+Q}{x} + \frac{(q-Q)\vec{y}\cdot\vec{x}}{2x^3} + \cdots\,.$$

This system has a monopole moment $q+Q$ and a dipole moment $\vec{p} = (q-Q)\vec{y}/2$.

3. (a) Show that the continuity equation for the charge density ρ and the current density $\vec{J}$ implies the conservation of charge in a volume V.

   (b) Derive $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$ starting from the integral form of the Biot-Savart law.

Solution (points: 6+7)

a) The continuity equation reads, $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{J} = 0$. Total charge in a volume V is given by $Q_V = \int_V d^3x\rho$. Then, for a fixed V, the continuity equation implies

$$\frac{\partial Q_V}{\partial t} = \int_V d^3x\frac{\partial\rho}{\partial t} = -\int_V d^3x\vec{\nabla}\cdot\vec{J} = -\oint_S d\vec{S}\cdot\vec{J} = -I_S\,,$$

where, I_S is the current flowing through the boundary S of the volume V, and we have used the divergence theorem. Thus, any change in the charge Q_V contained in volume V is entirely due to the flow of charge into or out of V, across the surface S. There is no creation or destruction of charge inside V.
b) The Biot-Savart law in integral form reads,

$$\vec{B}(\vec{x}) = \frac{1}{c}\int d^3x'\,\vec{J}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\,.$$

To get the result we need following relations,

$$\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|} = -\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}, \quad \nabla^2\frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\delta^3(\vec{x}-\vec{x}'),$$

and $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$. Note that $\vec{\nabla}$ involves differentiations with respect to $\vec{x}$ while $\vec{\nabla}'$ involves differentiations with respect to $\vec{x}'$. This difference should be kept in mind. Also we have to assume steady state so that $\vec{\nabla}\cdot\vec{J} = 0$. Then $\vec{B}(\vec{x}) = \frac{1}{c}\vec{\nabla}\times\int d^3x'\vec{J}(\vec{x}')/|\vec{x}-\vec{x}'|$ and

$$\begin{aligned}
\vec{\nabla}\times\vec{B}(\vec{x}) &= \frac{1}{c}\vec{\nabla}\times\left(\vec{\nabla}\times\int d^3x'\,\vec{J}(\vec{x}')/|\vec{x}-\vec{x}'|\right)\\
&= \frac{1}{c}\vec{\nabla}\left(\int d^3x'\,\vec{J}(\vec{x}')\cdot\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}\right) - \frac{1}{c}\int d^3x'\,\vec{J}(\vec{x}')\nabla^2\frac{1}{|\vec{x}-\vec{x}'|}\\
&= -\frac{1}{c}\vec{\nabla}\left(\int d^3x'\,\vec{J}(\vec{x}')\cdot\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}\right) + \frac{4\pi}{c}\vec{J}(\vec{x})\\
&= -\frac{1}{c}\vec{\nabla}\int d^3x'\left(\vec{\nabla}'\cdot\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} - \frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) + \frac{4\pi}{c}\vec{J}(\vec{x})\\
&= -\frac{1}{c}\vec{\nabla}\oint d\vec{S}'\cdot\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{4\pi}{c}\vec{J}(\vec{x})\,.
\end{aligned}$$

The first integral is a surface term and vanishes and hence the result.

4. (a) Suppose we demand that equations $\nabla\cdot\vec{E} = 4\pi\rho$ and $\nabla\cdot\vec{B} = 0$ are valid only at one instant of time $t = t_0$. Show that the remaining two Maxwell's equations then insure that the above equations remain valid at all times.

   (b) Consider the monochromatic plane-wave solutions of the source-free Maxwell equations in a medium, say, of the form $\vec{E}_0\cos(\vec{k}\cdot\vec{x}-\omega t)$ and $\vec{B}_0\cos(\vec{k}\cdot\vec{x}-\omega t)$ with $\omega/k = c/\sqrt{\epsilon\mu}$. Verify that the Poynting vector reproduces the expected energy current, $\vec{S} = \vec{v}\,u$, where u is the energy density of the electromagnetic field and $\vec{v}$ its propagation velocity.

Solution (points: 6+7)
a) Taking the divergences of $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$ and $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$ and using the continuity equation gives,

$$\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E} - 4\pi\rho) = 0\,, \qquad \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0\,,$$

which proves the result.

b) The energy density of $\vec{E}$ and $\vec{B}$ is $u = \frac{1}{8\pi}(\epsilon E^2 + \frac{1}{\mu}B^2)$. Demanding that the given solutions satisfy the Maxwell's equations implies that $\vec{E}$, $\vec{B}$ and $\vec{k}$ are all perpendicular to each other and also that $|\vec{B}| = \sqrt{\mu\epsilon}|\vec{E}|$. Hence for these solutions, $u = \frac{\epsilon}{4\pi}E^2$. On the other hand, in this case the Poynting vector becomes $\vec{S} \equiv \frac{c}{4\pi\mu}\vec{E}\times\vec{B} = \frac{c}{4\pi\mu}\sqrt{\mu\epsilon}\,E^2\,\hat{k}$. Expressed in terms of u, $\vec{S} = \frac{c}{\sqrt{\mu\epsilon}}u\,\hat{k}$. This is the expected result on realizing that $\frac{c}{\sqrt{\mu\epsilon}}\hat{k}$ is the propagation velocity of light in the medium.

5. (a) Starting with the wave equation for the vector potential $\vec{A}$ in Lorenz gauge and with a source $\vec{J}$, write the generic solution in terms of Green's function. Describe the physical meaning of *Advanced* and *Retarded* Green's functions (the Green's function need not be derived).

   (b) For a localized periodic source $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$, write down the solution in terms of the spherically symmetric retarded Green's function (derivation of the Green's function not needed). How does one characterize the Near, Intermediate and Far zones? Discuss the solution in the Far zone.

Solution (points: 7+7)

a) The wave equation for the vector potential in Lorenz gauge is

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J},$$

and the corresponding Green's function satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{x},t;\vec{x}',t') = -4\pi\delta(t-t')\delta^3(\vec{x}-\vec{x}').$$

The general solution is then given by

$$\vec{A}(\vec{x},t) = \vec{A}_0(\vec{x},t) + \int d^3x' \int dt'\, G(\vec{x},t;\vec{x}',t')\,\vec{J}(\vec{x}',t')\,,$$

where $\vec{A}_0$ is a solution of the homogeneous equation. The spherically symmetric solutions for the Green's function are,

$$\begin{aligned} G^+ &= \frac{1}{R}\delta(t'-t+R/c) \qquad \text{(Retarded Green's function)} \\ G^- &= \frac{1}{R}\delta(t'-t-R/c) \qquad \text{(Advanced Green's function)} \end{aligned}$$

where $R = |\vec{x}-\vec{x}'|$. If we interpret $(\vec{x}',t')$ as the specetime point where a disturbance is generated in the field by the source $\vec{J}(\vec{x}',t')$, and $(\vec{x},t)$ as the point where the disturbance is measured, then the retarded function implies that, $t = t' + R/c$. So the disturbance is detected a time interval R/c after it was created (consistent with causality). The advanced function implies that $t = t' - R/c$ so, in the above

conventions, *the disturbance was detected before it was created, violating causality.*
b) The solution in terms of the spherically symmetric retarded Green's function is

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \frac{\left[\vec{J}(\vec{x}',t')\right]_{ret}}{|\vec{x}-\vec{x}'|} ,$$

where the numerator is to be evaluated at the retarded time $t' = t - |\vec{x}-\vec{x}'|/c$. *Hence,* $\left[\vec{J}(\vec{x}',t')\right]_{ret} = \vec{J}(\vec{x}', t' = t - |\vec{x}-\vec{x}'|/c)$, *and for the given sinusoidal current,*

$$\vec{A}(\vec{x},t) = \frac{e^{-i\omega t}}{c}\int d^3x' \frac{\vec{J}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} ,$$

where $k = \omega/c$ *(*$= 2\pi/\lambda$*, say). There are three length scales in the problem: 1) the linear extent of the current distribution denoted by* d *(then, with the origin of the coordinate system chosen within the current distribution, one has* $x' \lesssim d$*), 2) the length* λ *which is the distance that a signal travels during one oscillation of the source (note that* $2\pi/\omega = T$ *is the time period of the oscillating source), 3) the distance to the observer denoted by* $x = |\vec{x}|$. *For a well localized source, we always assume that* $d << x, \lambda$. *Now, the space around the source may be divided into three different zones depending on the position of the observer relative to the "wavelength"* λ: *(i)* $d << x << \lambda$: *"near zone", (ii)* $d << x \sim \lambda$: *"intermediate zone" and (iii)* $d << \lambda << x$: *"far zone".*

In the far zone, since $d << x$ *in the exponent we can make the approximation* $|\vec{x}-\vec{x}'| \sim x - \hat{n}\cdot\vec{x}'$, *where* $\hat{n} = \vec{x}/x$ *(work this out). Also we use* $1/|\vec{x}-\vec{x}'| \sim 1/r$. *Then the solution becomes,*

$$\frac{1}{c}\frac{e^{i(kx-\omega t)}}{x}\int d^3x'\, \vec{J}(\vec{x}')\, e^{-ik\hat{n}\cdot\vec{x}'} .$$

Since x *is the radial distance form the center, this corresponds to spherical wave fronts moving out with speed* ω/k. *Hence in this region the disturbance behaves like radiation.*

6. (a) For a Lorentz transformation $\widetilde{x}^\mu = L^\mu{}_\nu x^\nu$, derive the transformation of *covariant* vectors.

   (b) Consider a frame $\widetilde{S}$ moving away from a frame S with velocity v in the x^1 direction. What fields $\vec{\widetilde{E}}$ and $\vec{\widetilde{B}}$ will the the $\widetilde{S}$ observer measure corresponding to an electric field in S?

   (c) Find the electric and magnetic fields produced by a charged particle in uniform motion, moving in the x^1 direction.

Solution (points: 5+5+4)
a) By definition, covariant vectors transform in the same way as the derivative operator ∂_μ. *Using the chain rule,* $\widetilde{\partial}_\mu = (\partial x^\nu/\partial\widetilde{x}^\mu)\partial_\nu$. *Inverting the Lorentz transformation given above yields ,* $x^\nu = (L^{-1})^\nu{}_\mu \widetilde{x}^\mu$, *and therefore,* $\partial x^\nu/\partial\widetilde{x}^\mu = (L^{-1})^\nu{}_\mu$.

Hence, under Lorentz transformations, covariant vectors transform as $\widetilde{V}_\mu(\widetilde{x}) = (L^{-1})^\nu{}_\mu V_\nu(x)$.

b) The Lorentz transformation of electric and magnetic fields $\vec{E}$ *and* $\vec{B}$ *are contained in* $\widetilde{F}^{\mu\nu}(\widetilde{x}) = L^\mu{}_\rho F^{\rho\sigma}(x)(L^T)_\sigma{}^\nu$, *or simply, in* $\widetilde{F}(\widetilde{x}) = L\,F(x)\,L^T$. *In this problem,*

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & 0 & 0 \\ E^2 & 0 & 0 & 0 \\ E^3 & 0 & 0 & 0 \end{pmatrix}.$$

Then carrying out the multiplication, one obtains the fields,

$$\begin{aligned} \widetilde{E}^1 &= E^1 & \qquad \widetilde{B}^1 &= 0 \\ \widetilde{E}^2 &= \gamma E^2 & \qquad \widetilde{B}^2 &= \gamma\beta E^3 \\ \widetilde{E}^3 &= \gamma E^3 & \qquad \widetilde{B}^3 &= -\gamma\beta E^2 \end{aligned}$$

c) To solve this, we the answer to part (b). Consider a reference frame S *moving along with the charged particle (and for convenience, centered at the location of the particle). In this frame, the particle is at rest and its electric field is given by* $\vec{E}(\vec{x}) = q\vec{x}/|x|^3$. *This is time independent and the magnetic field vanishes. Take* $\widetilde{S}$ *to be the frame of the observer in which the particle (and hence the frame* S*) moves with velocity* u *in the* x^1 *direction. Then* $\widetilde{S}$ *moves with respect to* S *with velocity* $v = -u$. *The fields observed by an observer in* $\widetilde{S}$ *are given in the answer to part (b) above with* $v = -u$. *But we have to express the* x^μ, *on which* $\vec{E}$ *depends, in terms of* $\widetilde{x}^\mu$. *That gives,* $x^1 = \gamma(\widetilde{x}^1 + \beta\widetilde{x}^0)$, *where* $\beta = v/c = -u/c$, *and*

$$|x|^3 \equiv [(x^1)^2 + (x^2)^2 + (x^3)^2]^{3/2} = [\gamma^2(\widetilde{x}^1 + \beta\widetilde{x}^0)^2 + (\widetilde{x}^2)^2 + (\widetilde{x}^3)^2]^{3/2}.$$

Explicitly,

$$\begin{aligned} \widetilde{E}^1 &= \gamma q\,\frac{(\widetilde{x}^1 - u\widetilde{x}^0/c)}{|x|^3}, & \qquad \widetilde{B}^1 &= 0, \\ \widetilde{E}^2 &= \gamma q\,\frac{\widetilde{x}^2}{|x|^3}, & \qquad \widetilde{B}^2 &= -\gamma uq\,\frac{\widetilde{x}^3}{c|x|^3}, \\ \widetilde{E}^3 &= \gamma q\,\frac{\widetilde{x}^3}{|x|^3}, & \qquad \widetilde{B}^3 &= \gamma uq\,\frac{\widetilde{x}^2}{c|x|^3}, \end{aligned}$$

where $|x|^3$ *is given above.*