Final Examination Paper for Electrodynamics-I

Date: *Tuesday, Jan 05, 2010*, Time: *09:00 - 15:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks*

`Note: Please explain your reasoning and calculations clearly`

Questions:	1	2	3	4	5	6	Total
Marks:	13	14	13	13	14	13	80

1. (a) Consider an electric field $\vec{E} = \hat{i}x + \hat{j}z + \hat{k}(f(x,y) + z^2)$. Determine $f(x,y)$ and compute the total charge contained in a cube specified by $0 \leq x, y, z \leq l$.

   (b) Derive the expression for the potential energy of a dipole in an electric field.

   Solution (points: 7+6)

   a) This is an electrostatic field with $E_x = x$, $E_y = z$, $E_z = f(x,y) + z^2$ and should satisfy $\vec{\nabla} \times \vec{E} = 0$. In terms of components of $\vec{E}$ this gives $\partial E_i/\partial x^j - \partial E_j/\partial x^i = 0$ for the indices i and j taking the values x, y, z, which, in turn, leads to $\partial f/\partial x = 0$ and $\partial f/\partial y - 1 = 0$. The unknown function $f(x,y)$ is therefore given by $f = y + c$ for an arbitrary constant c. So we have, $\vec{E} = \hat{i}x + \hat{j}z + \hat{k}(y + z^2 + c)$. From this, we can compute the charge density using $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ and get $\rho = (2z+1)/4\pi$. The total charge is then given by

$$Q = \frac{1}{4\pi}\int_0^l dx \int_0^l dy \int_0^l dz(1+2z) = \frac{1}{4\pi}(l^3 + l^4)$$

   (the total charge can also be computed using the Gauss law)

   b) Let us assume that the dipole is made of two charges q and $-q$ placed a small distance $\vec{l}$ apart, in the limit $\vec{l} \to 0$ keeping $\vec{p} = q\vec{l}$ fixed. In an electric field $\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x})$, the potential energy of the system, before taking the limit, is $q\Phi(\vec{x} + \frac{1}{2}\vec{l}) - q\Phi(\vec{x} - \frac{1}{2}\vec{l})$, where $\vec{x}$ is the position of the center of the charge system and $\vec{l}$ is taken to point from $-q$ to q. Using the Taylor expansion $\Phi(\vec{x}+\frac{1}{2}\vec{l}) = \Phi(\vec{x}) + \frac{1}{2}\vec{l}\cdot\vec{\nabla}\Phi(\vec{x}) + \cdots$ and taking the limit $\vec{l} \to 0$, one gets the dipole potential energy as $-\vec{p}\cdot\vec{E}(\vec{x})$.

2. Consider an external electric field given by $E_i = C_i + D_{ij}x^j$ in a region of space free of charges and currents.

   (a) Show that the matrix D_{ij} is traceless ($\sum_i D_{ii} = 0$) and symmetric ($D_{ij} = D_{ji}$). What is the external potential $\Phi_{ext}(\vec{x})$ corresponding to this electric field (use $\Phi_{ext}(0) = C_0$)?

   (b) In this external field place a *conducting* sphere of radius R cantered at $\vec{x} = 0$ and carrying zero net charge. Suppose the polarization of the sphere in the external field is described by a dipole moment p_i and a quadrupole moment Q_{ij}. Write the general expression for the induced potential $\Phi_{in}(\vec{x})$ for $|\vec{x}| \geq R$ generated by the multipole moments.

(c) Determine p_i and Q_{ij} in terms of C_i, D_{ij} and R and find the total potential $\Phi_{ext} + \Phi_{in}$ outside the sphere ($|\vec{x}| \geq R$).

(d) Compute the induced surface charge density on the sphere (Hint: If $\mathbf{e}_i$ denote the basis vectors in Cartesian coordinates, then $\vec{x} = x^i \mathbf{e}_i$ and for the radial unit vector, $\hat{x} = \vec{x}/|\vec{x}| = \hat{x}^i \mathbf{e}_i$. In spherical coordinates one can write, $x^i = x\hat{x}^i$ where $\hat{x}^3 = \cos\theta$, $\hat{x}^1 = \sin\theta\cos\phi$, $\hat{x}^2 = \sin\theta\sin\phi$. Hence, they do not vary with radial distance x).

Solution (points: 4+3+4+3)

a) The electric field satisfies $\vec{\nabla}\cdot\vec{E} = \sum_i \partial_i E^i = 0$ *implying* $\sum_i D_{ii} = 0$ *and* $(\vec{\nabla}\times\vec{E})_i = \sum_{jk} \epsilon_i{}^{jk}\partial_j E_k = 0$ *implying* $\sum_{jk}\epsilon_i{}^{jk} D_{jk} = 0$ *or* $D_{jk} = D_{kj}$. *Therefore, the matrix* D *is traceless and symmetric. The corresponding potential, consistent with* $\vec{E} = -\nabla\Phi_{ext}$, *is*

$$\Phi_{ext} = -\sum_i C_i x^i - \frac{1}{2}\sum_{ij} D_{ij}x^i x^j + C_0$$

b) For $|\vec{x}| \geq R$, *the induced potential due to the polarized sphere is the same as that due a dipole of moment* $\vec{p}$ *and a quadrupole of moment matrix* Q_{ij} *placed at the origin,*

$$\Phi_{in} = \frac{\vec{p}\cdot\vec{x}}{x^3} + \frac{1}{2}\frac{Q_{ij}x^i x^j}{x^5}$$

c) The sphere being conducting, the total potential $\Phi_{in} + \Phi_{ext}$ *on its surface must be constant*

$$\left(\frac{p_i x^i}{R^3} + \frac{1}{2}\frac{Q_{ij}x^i x^j}{R^5} - C_i x^i - \frac{1}{2}D_{ij}x^i x^j + C_0\right)\Bigg|_{|\vec{x}|=R} = const.$$

Since the x^i *vary on the surface, comparing terms with the same tensor structure, one gets* $p_i = R^3 C_i$ *and* $Q_{ij} = R^5 D_{ij}$. *Also the constant potential on the surface equals* C_0. *The total potential for* $|\vec{x}| \geq R$ *is then,*

$$\Phi = \Phi_{in} + \Phi_{ext} = C_i x^i\left(\frac{R^3}{x^3} - 1\right) + \frac{1}{2}D_{ij}x^i x^j\left(\frac{R^5}{x^5} - 1\right) + C_0$$

d) Using the notation described in the question, one can write the total potential ϕ *in spherical polar coordinates as*

$$\Phi = C_i\hat{x}^i\left(\frac{R^3}{x^2} - x\right) + \frac{1}{2}D_{ij}\hat{x}^i\hat{x}^j\left(\frac{R^5}{x^3} - x^2\right) + C_0$$

where $\hat{x}^i$ *are independent of* $x = |\vec{x}|$, *depending only on the angular variables. The surface charge density is given by* $(\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = 4\pi\sigma$ *where* $\hat{n}$ *is the unit normal to the surface of the sphere. In this case,* $\vec{E}_1 = 0$ *and* $\vec{E}_2\cdot\hat{n} = -\partial\Phi/\partial x|_{x=R}$. *Therefore,*

$$\sigma = -\frac{1}{4\pi}\frac{\partial\Phi}{\partial x}\Big|_{x=R} = \frac{1}{4\pi}\left(3C_i\hat{x}^i + \frac{5}{2}RD_{ij}\hat{x}^i\hat{x}^j\right)$$

3. (a) Using the expansion

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\, \frac{r'^{l}}{r^{l+1}}\, Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

develop the multipole expansion of the potential $\Phi(\vec{x})$ due to a localized charge distribution $\rho(\vec{x}')$ in terms of the multipole moments q_{lm} of ρ. Discuss how and under what conditions this expansion can be used to simplify a problem.

(b) Show that if the charge distribution has axial symmetry (that is, it is invariant under rotations about the z-axis), then the only non-zero multipole moments are q_{l0}.

(c) Using the above results, for two point charges q and $-q$ placed on the z-axis at $z = a$ and $z = -a$, compute the non-vanishing component of the dipole moment (given $Y_{10} = (\sqrt{3/4\pi})\cos\theta$).

Solution (points: 5+4+4)

a) The potential due to a localized charge distribution is given by

$$\Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}'|}$$

Using the expansion given in the question, it becomes,

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} \frac{4\pi}{2l+1}\, q_{lm}\, \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

This is the multipole expansion of the potential in terms of the multipole moments q_{lm} of the charge distribution given by

$$q_{lm} = \int d^3x' \rho(\vec{x}\,') r'^{l} Y^*_{lm}(\theta',\phi')$$

The multipole expansion allows us to parametrize the charge distribution in terms of its multipole moments. Further, the contribution of a moment q_{lm} to the potential falls off as $1/r^{l+1}$. Therefore, at large distances from a localized charge distribution, only a few non-zero multipole moments with the lowest values of l make significant contributions to Φ and are relevant. The remaining moments could be neglected. This allows us to parametrize even complicated charge distributions in terms of a few lowest l mutipole moments. The condition under which this approximation is valid is that the distance to the observation point (at which Φ is measured) is much larger as compared to the size of the charge distribution.

b) In the case of axial symmetry about the z-axis, ρ is independent of the azimuthal coordinate ϕ. In this case,

$$q_{lm} = (const)\left(\int_0^{\infty} r'^2 dr' \int_0^{\pi} \sin\theta' d\theta' \rho(r',\theta') r'^{l} P_l^m(\cos\theta')\right)\left(\int_0^{2\pi} d\phi' e^{-im\phi'}\right)$$

where we have used the fact that $Y_{lm}(\theta,\phi) = (const)P_l^m(\cos\theta)e^{im\phi}$. *Now, the* ϕ *integral gives a* δ_{m0}. *Thus the only non-vanishing moments in this case are* q_{l0}.

c) In this case the charge density is given by $\rho = q\delta(x')\delta(y')\left(\delta(z'-a) - \delta(z'+a)\right)$. *The three components of the dipole moment are* q_{1m}, *for* $m = 1, 0, -1$. *Since the problem has axial symmetry about the z-axis, the only non-vanishing component is* q_{10} *which is now given by (using the expressions for* Y_{10}, ρ *and noting that* $r' = \sqrt{x'^2 + y'^2 + z'^2}$*)*

$$\begin{aligned} q_{10} &= \int d^3x' \rho(\vec{x}\,')r'Y_{10}^*(\theta',\phi') = \sqrt{\frac{3}{4\pi}}\int dx' \int dy' \int dz' \\ &\quad \times q\delta(x')\delta(y')\left(\delta(z'-a) - \delta(z'+a)\right)\sqrt{x'^2+y'^2+z'^2}\cos\theta \\ &= \sqrt{\frac{3}{4\pi}}\int dz' q\left(\delta(z'-a) - \delta(z'+a)\right)|z'| = aq\sqrt{\frac{3}{\pi}} \end{aligned} \tag{1}$$

4. (a) Consider a current distribution with local current density $\vec{J}(\vec{x})$ in a volume V placed in an external magnetic field $\vec{B}(\vec{x})$. Find the expression for the net force $\vec{F}$ acting on the current distribution.

   (b) Work out the expression for the force acting between two thin wires carrying currents I_1 and I_2 (the wires are not necessarily straight and parallel).

   Solution (points: 6+7)
   a) The Lorentz force acting on a volume element d^3x *within the current distribution is* $d\vec{F} = \frac{1}{c}(\rho d^3x)\vec{v}\times\vec{B}$ *so that the force on the volume* V *is* $\vec{F} = \frac{1}{c}\int_V \vec{J}(\vec{x})\times\vec{B}(\vec{x})\, d^3x$
   b) For a wire we can write $d^3x = \vec{ds}\cdot\vec{dl}$ *with* $\vec{dl}$ *along the length of the wire and* $\vec{ds}$ *a surface element over the cross section of the wire. In the thin wire approximation,* $\vec{J}$ *is parallel to* $\vec{dl}$ *and the variation of* $\vec{B}$ *over the cross section can be neglected. Then,* $\int_V d^3x \vec{J}(\vec{x})\times\vec{B}(\vec{x}) = \int_l\int_S(\vec{ds}\cdot\vec{dl})(\vec{J}\times\vec{B}) = \int_l\int_S(\vec{ds}\cdot\vec{J})(\vec{dl}\times\vec{B}) = I_1\int_{l_1}\vec{dl}\times\vec{B}$. *This gives the force on wire 1 due to the field* $\vec{B}$. *Also in this approximation and using the same arguments, the magnetic field produced by a current* I_2 *becomes,* $\vec{B}(\vec{x}) = \frac{I_2}{c}\int_{l_2}\vec{dl}'\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$. *Using this in the force expression gives,*

$$F_{12} = \frac{I_1 I_2}{c^2}\int_{l_1}\int_{l_2}\vec{dl}\times\vec{dl}'\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

5. Consider a straight piece of length L and radius a of a long cylindrical wire which has resistance R and carries current I.

   (a) Find the electric and magnetic fields on the surface of the wire and indicate their directions.

   (b) Evaluate the energy carried into the wire by the above electric and magnetic fields.

(c) What happens to this energy in the steady state? Verify your answer using the Poynting theorem in the thin wire approximation.

Solution (points: 5+5+4)
a)The electric field on the surface is given by $E = \Phi/L$ *where the constant potential difference* Φ *is given by Ohm's law,* $\Phi = IR$*. Hence,* $E = IR/L$*. The direction of* $\vec{E}$ *is parallel to the current and hence to the wire. The magnetic field on the surface is given by Ampere's law as* $B = 2I/ca$ *(This is obtained by integrating* $\vec{\nabla}\times\vec{B} = (4\pi/c)\vec{J}$ *over a cross section of the wire and using the cylindrical symmetry of the problem). The direction of* $\vec{B}$ *is given by the "right-hand-rule" which makes it perpendicular to both* $\vec{E}$ *and the radius vector of the cylindrical wire. Hence the direction of* $\vec{B}$ *is along the angular direction of the cylinder.*
b) The energy carried into the wire by electric and magnetic fields is the surface integral of the Poynting vector over the surface of the wire. The Poynting vector is $\vec{S} = (c/4\pi)\vec{E}\times\vec{B}$*. Since* $\vec{E}$ *is perpendicular to* $\vec{B}$*, we have, for the magnitude of the Poynting vector,* $S = I^2R/(2\pi aL)$*.* $\vec{S}$ *is directed radially inward,* $\vec{S} = -\hat{r}\,S$*. The flux* $\int \vec{ds}\cdot\vec{S}$ *evaluated over the surface of a segment of length* L *of the cylindrical wire receives contributions only from the curved side-area of the cylinder (of area* $2\pi aL$*) and not from the top and bottom caps (since then* $\vec{S}$ *is perpendicular to* $\vec{ds}$*). Hence the total flux is*

$$\int \vec{ds}\cdot\vec{S} = -I^2R$$

The sign is due to fact that $\vec{S}$ *is directed radially inward while, for the cylinder,* $\vec{ds}$ *is directed radially outward. Physically, the negative sign signifies that energy enters into the volume under consideration, rather than leave it.*
c) Since in this problem the electric and magnetic fields are constant, the Poynting theorem reduces to

$$-\int_V d^3x\vec{E}\cdot\vec{J} = \int_{\partial V}\vec{ds}\cdot\vec{S}$$

The left hand side is recognized as the expression for the energy injected into the current distribution by the electric and magnetic fields. Thus the energy carried into the wire by the Poynting vector is fully converted into the kinetic energy of the charge carriers. Since the current is constant, the system is in steady state and the extra kinetic energy acquired by the charges is dissipated into heat as a result of collisions within the resistive medium. The left hand side can be computed in the thin wire approximation. For our wire, $d^3x = \vec{dl}\cdot\vec{ds}$ *where* $\vec{dl}$ *is along the length of the wire and the* $\vec{ds}$ *integration is over the cross sectional area of the wire. In the thin wire approximation,* $\vec{J}$ *is parallel to* $\vec{dl}$*, so that* $(\vec{dl}\cdot\vec{ds})(\vec{J}\cdot\vec{E}) = (\vec{J}\cdot\vec{ds})(\vec{dl}\cdot\vec{E})$*. Moreover,* $\vec{E}$ *can be taken to be constant over the cross section of the thin wire. Then*

$$-\int_V d^3x\vec{E}\cdot\vec{J} \approx -(\int\vec{J}\cdot\vec{ds})(\int\vec{dl}\cdot\vec{E}) = -I\Phi = -I^2R$$

which verifies the result of part b) of the question.

6. (a) For a coordinate transformation $x^\mu \to \widetilde{x}^\mu$ define *covariant* and *contravariant* vectors. For the special case of Lorentz transformations, show that if x^μ transforms as a contravariant vector, $\widetilde{x}^\mu = L^\mu{}_\nu x^\nu$, then $x_\mu = \eta_{\mu\nu} x^\nu$ transforms as a covariant vector.

(b) Consider,

$$m_0 \frac{dU^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu}\, U_\nu$$

where $U^\mu = (\gamma c, \gamma \vec{u})$ is the relativistic 4-velocity of a particle of charge q and rest mass m_0, τ is time in the rest-frame of the moving charge ($dt = \gamma d\tau$) and $\gamma^{-1} = \sqrt{1 - u^2/c^2}$.

i. Show that this contains the relativistic version of the Lorentz force law.

ii. What other information is contained in the above equation? Explain its physical significance.

Solution (points: 6+(7=4+3))

a) In general, covariant vectors W_μ and contravariant vectors V^μ transform as

$$\widetilde{W}_\mu(\widetilde{x}) = \frac{\partial x^\nu}{\partial \widetilde{x}^\mu} W_\nu(x)\,, \quad \widetilde{V}^\mu(\widetilde{x}) = \frac{\partial \widetilde{x}^\mu}{\partial x^\nu} V^\nu(x)$$

For the Lorentz transformation $\widetilde{x}^\mu = L^\mu{}_\nu x^\nu$, we get $\frac{\partial \widetilde{x}^\mu}{\partial x^\nu} = L^\mu{}_\nu$ and $\frac{\partial x^\nu}{\partial \widetilde{x}^\mu} = (L^{-1})^\nu{}_\mu$ so that $\widetilde{V}^\mu(\widetilde{x}) = L^\mu{}_\nu V^\nu(x)$ and $\widetilde{W}_\mu(\widetilde{x}) = (L^{-1})^\nu{}_\mu W_\nu(x)$ or, in matrix notation,

$$\widetilde{V} = LV\,, \qquad \widetilde{W} = (L^{-1})^T W$$

where V and W stand for 4-component column vectors. The defining equation for L, i.e., $L^T \eta L = \eta$ implies that $L^T = \eta L^{-1} \eta^{-1}$ or $(L^T)^{-1} = \eta L \eta^{-1}$. Using this, the Lorentz transformation of a covariant vector becomes, in matrix notation,

$$\widetilde{W} = \eta L \eta^{-1} W$$

Let us now look at the Lorentz transformation of $\eta_{\mu\nu} x^\nu$. In matrix notation, the Lorentz transformed quantity is

$$\eta \widetilde{x} = \eta L x = \eta L \eta^{-1} (\eta x)$$

which is the Lorentz transformation of a covariant vector.

b)(i) For $\mu = i$, the relativistic equation reduces to

$$m \frac{du^i}{dt} + \frac{m}{\gamma} \frac{d\gamma}{dt} u^i = qF^{i0} - \frac{q}{c} F^{ij}\, u_j$$

But, $F^{i0} = E^i$, $F^{ij} = -\epsilon^{ij}{}_k B^k$ and $(\vec{u} \times \vec{B})^i = -\epsilon^i{}_{jk} u^j B^k$ (with $\epsilon_{123} = 1$), and therefore,

$$m \frac{d\vec{u}}{dt} + \frac{m}{\gamma} \frac{d\gamma}{dt} \vec{u} = q\vec{E} + \frac{q}{c} \vec{u} \times \vec{B}$$

where $m = \gamma m_0$. *In terms of the momentum* $\vec{p} = m_0\gamma\vec{u}$, *it becomes,*

$$\frac{d\vec{p}}{dt} = q\vec{E} + \frac{q}{c}\vec{u} \times \vec{B}$$

Note that γ *being a function of the velocity* $\vec{u}$ *of the moving particle, is not constant in time.*
(ii) For $\mu = 0$, *it reduces to* $d(m_0c^2\gamma)/dt = qE^iu^i$. *We recognize* $\mathcal{E} = m_0c^2\gamma$ *as the relativistic energy of the particle. Hence* $d(\mathcal{E})/dt = q\vec{u} \cdot \vec{E}$ *which gives the power transferred to the charged particle from the electric field.*