Final Examination Paper for Electrodynamics-I

Date: *Friday, Oct 30, 2009*, Time: *09:00 - 15:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks or equivalent*

`Note: Please explain your reasoning and calculations clearly`

Questions:	1	2	3	4	5	6	Total
Marks:	13	13	13	13	14	14	80

1. (a) Consider the Poisson equation, $\nabla^2\phi = -4\pi\rho$, and the corresponding Green's function equation in a volume V with a boundary S. On S, ϕ satisfies either Neumann or Dirichlet boundary conditions. Derive the general solution for ϕ in terms of the Green's function for both boundary conditions.

Solution (points: 13)

The solution is outlined here (for details see Jackson's section 1.8 and 1.10, 3rd edition): The Green's function equation corresponding to the Laplace equation is

$$\nabla^2 G(\vec{x}-\vec{x}\,') = -4\pi\delta(\vec{x}-\vec{x}\,') \;\Rightarrow\; G(\vec{x}-\vec{x}\,') = \frac{1}{|\vec{x}-\vec{x}\,'|} + F(\vec{x},\vec{x}\,')$$

where $\nabla^2 F(\vec{x},\vec{x}\,') = 0$ within the volume V and F is chosen such that G satisfies specified boundary conditions on the boundary S of V. Now, in Green's second identity,

$$\int_V d^3x'(\psi_1\nabla'^2\psi_2 - \psi_2\nabla'^2\psi_1) = \oint_S ds'(\psi_1\frac{\partial\psi_2}{\partial n'} - \psi_2\frac{\partial\psi_1}{\partial n'})$$

choose $\psi_1(\vec{x}\,') = \phi(\vec{x}\,')$ and $\psi_2(\vec{x}\,') = G(\vec{x}-\vec{x}\,')$ to get the formal solution to the Laplace equation,

$$\phi(\vec{x}) = \int_V d^3x'\rho(\vec{x}\,')G(\vec{x}-\vec{x}\,') + \frac{1}{4\pi}\oint_S ds'\left[G(\vec{x}-\vec{x}\,')\frac{\partial\phi}{\partial n'}(\vec{x}\,') - \phi(\vec{x}\,')\frac{\partial}{\partial n'}G(\vec{x}-\vec{x}\,')\right]$$

For Dirichlet boundary conditions on ϕ, we are given $\phi(\vec{x}\,')$ for all $\vec{x}\,'$ on the surface S. Then on G we need the Dirichlet boundary conditions

$$G_D(\vec{x}-\vec{x}\,') = 0, \quad for\ all \quad \vec{x}\,'\ on\ S$$

leading to

$$\phi(\vec{x}) = \int_V d^3x'\rho(\vec{x}\,')G_D(\vec{x}-\vec{x}\,') - \frac{1}{4\pi}\oint_S ds'\left[\phi(\vec{x}\,')\frac{\partial}{\partial n'}G_D(\vec{x}-\vec{x}\,')\right]$$

For Neumann boundary conditions on ϕ, we are given $\frac{\partial\phi}{\partial n'}(\vec{x}\,')$ for all $\vec{x}\,'$ on S. However, we cannot simply impose $\frac{\partial}{\partial n'}G(\vec{x}-\vec{x}\,') = 0$ since this contradicts the defining equation for G. Then the simplest boundary condition is

$$\frac{\partial}{\partial n'}G(\vec{x}-\vec{x}\,') = -\frac{4\pi}{A}, \quad for\ all \quad \vec{x}\,'\ on\ S$$

where A is the total area of the boundary S. This leads to the solution

$$\phi(\vec{x}) = \langle\phi\rangle_S + \int_V d^3x' \rho(\vec{x}') G_N(\vec{x}-\vec{x}') + \frac{1}{4\pi}\oint_S ds' \left[G_N(\vec{x}-\vec{x}')\frac{\partial\phi}{\partial n'}(\vec{x}') \right]$$

where $\langle\phi\rangle_S$ is average value of ϕ over the surface S.

2. (a) Consider a charge q placed a distance d in front of an infinite plane conductor kept at zero potential. Determine the potential $\phi(\vec{x})$ at any point in front of the conductor using the method of images.

   (b) Consider a macroscopic volume V within a polarized dielectric material containing free charges of density $\rho_f(\vec{x}\,')$ and a *dipole moment density* $\vec{P}(\vec{x}\,')$. Write the expression for the electric potential $\phi(\vec{x})$ at $\vec{x}$ due to free charges and dipoles within V and show that the effect of the Polarization can be undersood in term of a polarization charge density ρ_{pol}.

Solution (points: 7+6)

a) For simplicity, choose a coordinate system such that the z-axis is perpendicular to the conducting plane and the charge q lies on the z-axis with position vector $d\hat{z}$. By the symmetry of the problem, the image charge q' should also be located on the z-axis at some position $-\hat{z}d'$. Then, at some point $\vec{x}$ with respect to the origin of our coordinate system, the potential is given by

$$\phi(\vec{x}) = \frac{q}{|\vec{x}-\hat{z}d|} + \frac{q'}{|\vec{x}+\hat{z}d'|}$$

For points $\vec{x}$ on the conducting surface, $\phi(\vec{x}|_{surface}) = 0$, but also , $\vec{x}\cdot\hat{z} = 0$. Hence,

$$0 = \frac{q}{\sqrt{|x^2+d^2|}} + \frac{q'}{\sqrt{|x^2+d'^2|}}$$

and it should hold for all values of $x = |\vec{x}|$. Simple manipulations then show that $d' = d$ and $q' = -q$. The potential at any point is then given by

$$\phi(\vec{x}) = \frac{q}{|\vec{x}-\hat{z}d|} - \frac{q}{|\vec{x}+\hat{z}d|}$$

b) Start with the expression for the electrostatic potential at $\vec{x}$ due to a dipole moment density in a volume ΔV around a point $\vec{x}'$, $\Delta V \vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')/|\vec{x}-\vec{x}'|^3$. Then the total potential at $\vec{x}$ is

$$\Phi(\vec{x}) = \int_V d^3x' \left(\frac{\rho_f(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right)$$

The second term becomes

$$\int_V d^3x' \vec{P}(\vec{x}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -\int_V d^3x' \frac{\vec{\nabla}'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

where we have dropped a surface term arising from an integration by parts. Now, using $\nabla^2_{(x)}(1/|\vec{x}-\vec{x}'|) = -4\pi\delta(\vec{x}-\vec{x}')$*, one gets*

$$\vec{\nabla}\cdot\vec{E} = 4\pi(\rho_f - \vec{\nabla}\cdot\vec{P})$$

Thus, we see that $-\nabla\cdot\vec{P}$ *can be regarded as an effective charge density due to the polarization of the medium.*

3. (a) Consider moving charges giving rise to a current density $\vec{J}$ within a volume V in the presence of electric and magnetic fields. Show that the total power injected into the current distribution by the fields is given by $\int_V d^3x\,\vec{J}\cdot\vec{E}$.

   (b) Using Maxwell's equations, derive the *Poynting theorem* [You may need the vector identity $\nabla\cdot(\vec{P}\times\vec{Q}) = (\nabla\times\vec{P})\cdot\vec{Q} - \vec{P}\cdot(\nabla\times\vec{Q})$].

   (c) Give the physical interpretation of each term in the mathematical expression for the Poynting theorem. What is the physical meaning of the Poynting theorem?

Solution (points: 4+5+4)
a) The power transferred to a point charge q on which a force $\vec{F}$ *acts is the rate of change of its kinetic energy,* $\frac{1}{2}mv^2$*, that is,* $d(\frac{1}{2}mv^2)/dt = \vec{F}\cdot\vec{v}$*. Using the Lorentz force law and* $\vec{v}\cdot(\vec{v}\times\vec{B}) = 0$*, this becomes* $q\vec{v}\cdot\vec{E}$*. For charges contained in volume* d^3x *within a continuous charge distribution, one has* $q \to \rho d^3x$*. Using* $\vec{J} = \rho\vec{v}$ *and integrating over the volume of the current distribution, leads to the desired result.*

b) To obtain the Poynting theorem, start with $\vec{J}\cdot\vec{E}$ *and, using Maxwell's equations, rewrite* $\vec{J}$ *in terms of* $\vec{E}$ *and* $\vec{B}$*. After some manipulations, one gets,*

$$\vec{J}\cdot\vec{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon\vec{E}\cdot\vec{E} + \frac{1}{\mu}\vec{B}\cdot\vec{B}\right) + \frac{c}{4\pi}\nabla\cdot(\vec{E}\times\vec{H}) = 0$$

c) $\vec{J}\cdot\vec{E}$*: power injected into the current distribution by the electric field/unit volume.*
$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon\vec{E}\cdot\vec{E} + \frac{1}{\mu}\vec{B}\cdot\vec{B}\right)$*: Rate of change of energy densities of the electric and magnetic fields.*
$\frac{c}{4\pi}\nabla\cdot(\vec{E}\times\vec{H})$*: Energy flux per unit time per unit volume carried by the electromagnetic fields.* $\frac{c}{4\pi}(\vec{E}\times\vec{H})$ *is the Poynting vector that corresponds to the energy flux per unit area per unit time across a surface as follows from the divergence theorem,* $\int_V d^3x\nabla\cdot(\vec{E}\times\vec{H}) = \int_S d\vec{S}\cdot(\vec{E}\times\vec{H})$*. The Poynting theorem is a statement of conservation of energy and also provides a mathematical expression for the energy carried by electromagnetic waves in the form of the Poynting vector.*

4. (a) A current described by density $\vec{J}$ flows within a volume V in a magnetic field $\vec{B}$. Compute the total force acting on the current distribution. Derive the corresponding expression for a thin wire carrying total current I.

   (b) Consider Maxwell's equations in free space and in the absence of sources. For solutions with space-time dependence given by $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, show that $\vec{E}$, $\vec{B}$ and the wave vector $\vec{k}$ are all perpendicular to each other.

Solution (points: 7+6)

a) The Lorentz force acting on a volume element d^3x within the current distribution is $d\vec{F} = \dfrac{1}{c}(\rho d^3x)\vec{v}\times\vec{B}$ so that the force on the volume V is $\vec{F} = \frac{1}{c}\int_V \vec{J}(\vec{x})\times\vec{B}(\vec{x})\, d^3x$. For a wire we can write $d^3x = \vec{ds}\cdot\vec{dl}$ with $\vec{dl}$ along the length of the wire and $\vec{ds}$ a surface element over the cross section of the wire. In the thin wire approximation, $\vec{J}$ is parallel to $\vec{dl}$ and the variation of $\vec{B}$ over the cross section can be neglected. Then, $\int_V d^3x\vec{J}(\vec{x})\times\vec{B}(\vec{x}) = \int_l\int_S(\vec{ds}\cdot\vec{dl})(\vec{J}\times\vec{B}) = \int_l\int_S(\vec{ds}\cdot\vec{J})(\vec{dl}\times\vec{B}) = I\int_l \vec{dl}\times\vec{B}$.

b) The given solutions are of the type,

$$\vec{E}(\vec{x},t) = \vec{E}_{\rm o}e^{i(\vec{k}\cdot\vec{x}-\omega t)}\,, \qquad \vec{B}(\vec{x},t) = \vec{B}_{\rm o}e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

which give $\vec{\nabla}\cdot\vec{E} = i\vec{k}\cdot\vec{E}$ (and similarly for $\vec{B}$). Then $\vec{\nabla}\cdot\vec{E}=0$ and $\vec{\nabla}\cdot\vec{B}=0$ imply $\vec{k}\cdot\vec{E}=0$, $\vec{k}\cdot\vec{B}=0$. Now substituting the solutions in the remaining two Maxwell's equations gives $\vec{k}\times\vec{E} = \dfrac{\omega}{c}\vec{B}$ and $\vec{k}\times\vec{B} = -\dfrac{\omega}{c}\vec{E}$ which proves that $\vec{E}$ and $\vec{B}$ are also perpendicular to each other.

5. (a) Suppose we demand that equations $\nabla\cdot\vec{E} = 4\pi\rho$ and $\nabla\cdot\vec{B}=0$ are valid only at one instant of time $t=t_0$. Show that the remainig two Maxwell's equations then insure that the above equations remain valid at all times.

   (b) Starting with the wave equation for the vector potential $\vec{A}$ in Lorenz gauge, write down the solution in terms of the spherically symmetric retarded Green's function (the Green's function need not be derived) for a localized source $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$. How does one characterize the Near, Intermediate and Far zones? Discuss the solution in the Near zone.

Solution (points: 5+9)

a) Taking the divergences of $\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$ and $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$ and using the continuity equation gives,

$$\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E} - 4\pi\rho) = 0\,, \qquad \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0$$

which proves the result.

b) The wave equation for the vector potential in Lorenz gauge is

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}$$

The solution in terms of the spherically symmetric retarded Green's function is

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x'\frac{\left[\vec{J}(\vec{x}',t')\right]_{ret}}{|\vec{x}-\vec{x}'|}$$

where the numerator is to be evaluatred at the retarded time $t' = t - |\vec{x}-\vec{x}'|/c$. Hence, $\left[\vec{J}(\vec{x}',t')\right]_{ret} = \vec{J}(\vec{x}', t' = t - |\vec{x}-\vec{x}'|/c)$, and for the given sinusoidal current,

$$\vec{A}(\vec{x},t) = \frac{e^{-i\omega t}}{c}\int d^3x'\frac{\vec{J}(\vec{x}')\,e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

where $k = \omega/c$ $(= 2\pi/\lambda$, say). There are three length scales in the problem: 1) the linear extension of the current distribution denoted by d (then, with the origin of the coordinate system chosen within the current distribution, one has $x' \lesssim d$), 2) the length λ which is the distance that a signal travels during one oscillation of the source (note that $2\pi/\omega = T$ is the time period of the oscillating source), 3) the distance to the observer denoted by $x = |\vec{x}|$. For a well localized source, we always assume that $d << x, \lambda$. Now, the space around the source may be devided into three different zones depending on the position of the observer relative to the "wavelength" λ: (i) $d << x << \lambda$: "near zone", (ii) $d << x \sim \lambda$: "intermediate zone" and (iii) $d << \lambda << x$: "far zone". In the near zone, we can make the approximation $k|\vec{x} - \vec{x}'| \sim k|\vec{x}| << 1$ or $e^{ik|\vec{x}-\vec{x}'|} \sim 1$, so that

$$\vec{A}(\vec{x}, t) = \frac{e^{-i\omega t}}{c} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

Except for the overall time modulation, this has the character of a magnetostatic field.

6. (a) i. Verify that the two sourced Maxwell's equations are contained in the relativistic equation $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$.

   ii. Show that this leads to the continuity equation in covariant form.

   iii. Further, show that the relativistic equations of part (i) and (ii) have the same form in all inertial reference frames.

   (b) Assume that an inertial reference frame $\tilde{S}$ is moving away from a frame S with velocity v in the positive x^1 direction. If the observer in S measures fields corresponding to an electrostatic *potential* $\phi(\vec{x}) = Q/x$, where $x = \sqrt{\sum_1^3 x^i x^i}$, find the electric and magnetic *potentials* as measured by the observer in $\tilde{S}$. Interpret your result in physical terms.

Solution (points: 7+7)

a) (i) We start by writing the equation with the source J^ν seperately for $\nu = 0$ and $\nu = j$ (where j is a space index). The index μ is summed over so all its values are retained,

$$\partial_i F^{i0} = \frac{4\pi}{c} J^0 \qquad \partial_0 F^{0j} + \partial_i F^{ij} = \frac{4\pi}{c} J^j$$

where we have used $F^{00} = 0$. Now we note that $J^0 = c\rho$, $F^{i0} = E^i$, $F^{ij} = -\epsilon^{ij}{}_k B^k$ and $\partial_i F^{ij} = -\epsilon^{ij}{}_k \partial_i B^k = (\vec{\nabla} \times \vec{B})^j$. Thus, we recover the two sourced Maxwell equations (you may avoid using ϵ_{ijk} and write F_{12}, etc., directly in terms of B_i),

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho\,, \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{J}$$

(ii) Since ∂_μ and ∂_ν commute and $F^{\mu\nu} = -F^{\nu\mu}$ we have $\partial_\mu \partial_\nu F^{\mu\nu} = 0$. Hence the given relativistic equation leads to $\partial_\mu J^\mu = 0$.

(iii) In a different Lorentz frame, the expressions $\partial_\mu F^{\mu\nu} - \frac{4\pi}{c} J^\nu$ and $\partial_\mu J^\mu$ take the form

$$\tilde{\partial}_{\tilde{\mu}} \tilde{F}^{\tilde{\mu}\tilde{\nu}} - \frac{4\pi}{c} \tilde{J}^{\tilde{\nu}} = L^{\tilde{\nu}}{}_{\nu} (\partial_\mu F^{\mu\nu} - \frac{4\pi}{c} J^\nu)$$

and $\tilde{\partial}_\mu \tilde{J}^\mu = \partial_\mu J^\mu$, where $L^{\tilde{\mu}}{}_{\mu}$ are the components of the Lorentz transformation matrix. Thus the expressions have the same form in the two frames and when the equations are satisfied in the original frame, the corresponding equations in transformed frame also hold (since L is invertible).
b) The electric potential $\phi(x)$ and magnetic potential $\vec{A}(x)$ combine into a 4-vector $A^\mu = \{A^0 = \phi, \vec{A}\}$ which under Lorentz transformations L transforms as

$$\tilde{A}^\mu(\tilde{x}) = L^\mu_\nu A^\nu (L^{-1}\tilde{x})$$

In our case, $\vec{A} = 0$ and the non-trivial components of L are, $L^0_{\ 0} = L^1_{\ 1} = \gamma$, and $L^1_{\ 0} = L^0_{\ 1} = -\gamma\beta$. Therefore Lorentz transformation gives (suppressing the $\tilde{x}$ dependence)

$$\tilde{\phi} = \gamma\phi\,, \qquad \tilde{A}^1 = -\gamma\beta\phi\,, \qquad \tilde{A}^2 = \tilde{A}^3 = 0$$

To complete the transformation, we have to express the x^μ dependence of ϕ in terms of $\tilde{x}^\mu$. For the given Lorentz transformation, $x^1 = \gamma(\tilde{x}^1 + \beta\tilde{x}^0)$, $x^2 = \tilde{x}^2$ and $x^3 = \tilde{x}^3$, so that $x^2 = \sum_1^3 x^i x^i = \gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2$. Then,

$$\tilde{\phi}(\tilde{x}) = \gamma \frac{Q}{\sqrt{(\gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2)}}\,, \qquad \tilde{A}^1 = -\beta\tilde{\phi}(\tilde{x})$$

Physical interpretation: Ignoring relativistic effects, simply because of the relative motion, the stationary charge Q at the origin of S appears, to the $\tilde{S}$ observer, as a moving charge Q at a varying x_1 distance $\tilde{x}_1 + v\tilde{t}$, giving rise to a current $-\vec{v}Q$. Hence it gives rise to both electric and magnetic fields. Factors of γ take care of relativistic effects.