# Final Examination Paper for Electrodynamics-I

Date: *Saturday, Jan 03, 2009*, Time: *09:00 - 15:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks or equivalent*

`Note: Please explain your reasoning and calculations clearly`

| Questions: | 1 | 2 | 3 | 4 | 5 | 6 | Total |
|---|---|---|---|---|---|---|---|
| Marks: | 13 | 13 | 13 | 13 | 14 | 14 | 80 |

1. (a) Consider a charge $q$ placed a distance $d$ in front of an infinite plane conductor kept at zero potential. Determine the potential $\Phi(\vec{x})$ at any point in front of the conductor using the method of images.

   (b) Consider the Poisson equation, $\nabla^2\phi = -4\pi\rho$, in a volume $V$ with a boundary $S$ over which $\Phi$ satisfies either Neumann or Dirichlet boundary conditions. Write down the corresponding Green's function equation and express the solutions $\Phi$ in terms of the Neumann or Dirichlet Green's functions.

**Solution** (points: 13)

*a) For simplicity, choose a coordinate system such that the z-axis is perpendicular to the conducting plane and the charge q lies on the z-axis with position vector $d\hat{z}$. By the symmetry of the problem, the image charge $q'$ should also be located on the z-axis at some position $-\hat{z}d'$. Then, at some point $\vec{x}$ with respect to the origin of our coordinate system, the potential is given by*

$$\phi(\vec{x}) = \frac{q}{|\vec{x} - \hat{z}d|} + \frac{q'}{|\vec{x} + \hat{z}d'|}\,.$$

*For points $\vec{x}$ on the conducting surface, $\phi(\vec{x}|_{surface}) = 0$, but also , $\vec{x}\cdot\hat{z} = 0$. Hence,*

$$0 = \frac{q}{\sqrt{|x^2 + d^2|}} + \frac{q'}{\sqrt{|x^2 + d'^2|}}\,,$$

*and it should hold for all values of $x = |\vec{x}|$. Simple manipulations then show that $d' = d$ and $q' = -q$. The potential at any point is then given by*

$$\phi(\vec{x}) = \frac{q}{|\vec{x} - \hat{z}d|} - \frac{q}{|\vec{x} + \hat{z}d|}\,.$$

*b) The solution is outlined here (for details see Jackson's section 1.8 and 1.10, $3^{rd}$ edition): The Green's function equation corresponding to the Laplace equation is*

$$\nabla^2 G(\vec{x} - \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}') \;\Rightarrow\; G(\vec{x} - \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')\,,$$

*where* $\nabla^2 F(\vec{x}, \vec{x}') = 0$ *within the volume* $V$ *and* $F$ *is chosen such that* $G$ *satisfies specified boundary conditions on the boundary* $S$ *of* $V$*. Now, in Green's second identity,*

$$\int_V d^3x'(\psi_1 \nabla'^2 \psi_2 - \psi_2 \nabla'^2 \psi_1) = \oint_S ds'(\psi_1 \frac{\partial \psi_2}{\partial n'} - \psi_2 \frac{\partial \psi_1}{\partial n'}) \,,$$

*choose* $\psi_1(\vec{x}') = \phi(\vec{x}')$ *and* $\psi_2(\vec{x}') = G(\vec{x} - \vec{x}')$ *to get the formal solution to the Laplace equation,*

$$\phi(\vec{x}) = \int_V d^3x' \rho(\vec{x}') G(\vec{x} - \vec{x}') + \frac{1}{4\pi} \oint_S ds' \left[ G(\vec{x} - \vec{x}') \frac{\partial \phi}{\partial n'}(\vec{x}') - \phi(\vec{x}') \frac{\partial}{\partial n'} G(\vec{x} - \vec{x}') \right] .$$

*For Dirichlet boundary conditions on* $\phi$*, we are given* $\phi(\vec{x}')$ *for all* $\vec{x}'$ *on the surface* $S$*. Then on* $G$ *we need the Dirichlet boundary conditions*

$$G_D(\vec{x} - \vec{x}') = 0, \quad \textit{for all} \quad \vec{x}' \textit{ on } S \,,$$

*leading to*

$$\phi(\vec{x}) = \int_V d^3x' \rho(\vec{x}') G_D(\vec{x} - \vec{x}') - \frac{1}{4\pi} \oint_S ds' \left[ \phi(\vec{x}') \frac{\partial}{\partial n'} G_D(\vec{x} - \vec{x}') \right] .$$

*For Neumann boundary conditions on* $\phi$*, we are given* $\frac{\partial \phi}{\partial n'}(\vec{x}')$ *for all* $\vec{x}'$ *on* $S$*. However, we cannot simply impose* $\frac{\partial}{\partial n'} G(\vec{x} - \vec{x}') = 0$ *since this contradicts the defining equation for* $G$*. Then the simplest boundary condition is*

$$\frac{\partial}{\partial n'} G(\vec{x} - \vec{x}') = -\frac{4\pi}{A}, \quad \textit{for all} \quad \vec{x}' \textit{ on } S \,,$$

*where* $A$ *is the total area of the boundary S. This leads to the solution*

$$\phi(\vec{x}) = \langle \phi \rangle_S + \int_V d^3x' \rho(\vec{x}') G_N(\vec{x} - \vec{x}') + \frac{1}{4\pi} \oint_S ds' \left[ G_N(\vec{x} - \vec{x}') \frac{\partial \phi}{\partial n'}(\vec{x}') \right] ,$$

*where* $\langle \phi \rangle_S$ *is average value of* $\phi$ *over the surface S.*

2. (a) Consider an electric dipole of moment $\vec{p}$ placed at the origin. Evaluate the electrostatic potential at a point $\vec{x}$ generated by this dipole.

   (b) Consider a macroscopic volume $V$ within a polarized dielectric material containing free charges of density $\rho_f(\vec{x}\,')$ and a *dipole moment density* $\vec{P}(\vec{x}\,')$. Compute the electric potential $\phi(\vec{x})$ due to free charges and dipoles within V. Express polarization charge density $\rho_{pol}$ and the *electric displacement* vector $\vec{D}$ in terms of $\vec{P}$.

**Solution** (points: 13)

*a) A dipole is constructed from charges* $q$ *and* $-q$ *a distance* $\vec{y}$ *apart. In the setup given, the midpoint of* $\vec{y}$ *is the origin of the coordinate system. Then the potential at any point* $\vec{x}$ *is given by,*

$$\Phi(\vec{x}) = \frac{q}{|\vec{x} - \vec{y}/2|} - \frac{q}{|\vec{x} + \vec{y}/2|} \,.$$

*Expanding in powers of $1/x$ (where $x = |\vec{x}|$),*

$$\frac{1}{|\vec{x} \mp \vec{y}/2|} = \frac{1}{x} \pm \frac{\vec{y} \cdot \vec{x}}{2x^3} + \cdots .$$

*Then, in the limit $\vec{y} \to 0$, keeping $\vec{p} = q\vec{y}$ fixed, we obtain the dipole potential*

$$\Phi(\vec{x}) = \frac{\vec{p} \cdot \vec{x}}{x^3} .$$

*b) In a medium of dipole moment density $\vec{P}(\vec{x})$, the dipole moment in volume $\Delta V$ around a point $\vec{x}'$ is $\Delta V \vec{P}(\vec{x}')$. This contributes to the electrostatic potential at $\vec{x}$ as $\Delta V \vec{P}(\vec{x}') \cdot (\vec{x} - \vec{x}')/|\vec{x} - \vec{x}'|^3$. Then the total potential at $\vec{x}$ is*

$$\Phi(\vec{x}) = \int_V d^3x' \left( \frac{\rho_f(\vec{x}')}{|\vec{x} - \vec{x}'|} + \frac{\vec{P}(\vec{x}') \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \right) .$$

*The second term becomes*

$$\int_V d^3x' \vec{P}(\vec{x}') \cdot \vec{\nabla}'(\frac{1}{|\vec{x} - \vec{x}'|}) = - \int_V d^3x' \frac{\vec{\nabla}' \cdot \vec{P}(\vec{x}')}{|\vec{x} - \vec{x}'|} ,$$

*where we have dropped a surface term arising from an integration by parts. Now, using $\nabla^2_{(x)}(1/|\vec{x} - \vec{x}'|) = -4\pi\delta(\vec{x} - \vec{x}')$, one gets*

$$\vec{\nabla} \cdot \vec{E} = 4\pi(\rho_f - \vec{\nabla} \cdot \vec{P}) .$$

*Thus, we see that $-\nabla \cdot \vec{P} = \rho_{pol}$ can be regarded as an effective charge density due to the polarization of the medium. The above equation can also be rewritten as $\vec{\nabla} \cdot (\vec{E} + 4\pi\vec{P}) = 4\pi\rho_f$. In terms of the electric displacement vector $\vec{D} = \vec{E} + 4\pi\vec{P}$, this becomes $\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f$.*

3. (a) Consider a length $l$ of thin conducting wire, through which a current $I$ flows, placed in a magnetic field $\vec{B}$. Derive the expression for the force that acts on the wire segment. For a rectangular wire loop of height $L$ and width $w$ placed in a uniform magnetic field which is perpendicular to the sides of length $L$, compute the net force and torque acting on the the wire loop.

   (b) Show that, in time dependent situations, the equation $\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{J}$ is not consistent with charge conservation. Derive the modified equation by requiring consistency with charge conservation.

**Solution** (points: 13)

*a) The Lorentz force acting on a volume element $d^3x$ within the current distribution is $d\vec{F} = \dfrac{1}{c}(\rho d^3x)\vec{v} \times \vec{B}$ so that the force on the volume $V$ is $\vec{F} = \frac{1}{c} \int_V \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x$. For a wire we can write $d^3x = \vec{ds} \cdot \vec{dl}$ with $\vec{dl}$ along the length of the wire and $\vec{ds}$ a surface element over the cross section of the wire. In the thin wire approximation,*

*$\vec{J}$ is parallel to $\vec{dl}$ and the variation of $\vec{B}$ over the cross section can be neglected. Then, $\int_V d^3x \vec{J}(\vec{x}) \times \vec{B}(\vec{x}) = \int_l \int_S (\vec{ds} \cdot \vec{dl})(\vec{J} \times \vec{B}) = \int_l \int_S (\vec{ds} \cdot \vec{J})(\vec{dl} \times \vec{B})$. This gives the force on wire $l$ due to the magnetic field $\vec{B}$ as,*

$$\vec{F} = \frac{I}{c} \int_l \vec{dl} \times \vec{B} \,.$$

*Now consider a rectangular current loop with its height $L$ parallel to the $z$-axis, placed in a uniform magnetic field $\vec{B}$ in the $\hat{y}$ direction. Clearly the loop edges of length $w$ are in the $x-y$ plane. Let $\theta$ denote the angle between the vector normal to the loop area and the magnetic field $\vec{B}$ along $\hat{y}$. Clearly this also is the angle between the edge of length $w$ and the $x$-axis. Now using the above formula, the forces on the two edges $L\hat{z}$ and $-L\hat{z}$ are $\mp(I/c)LB\hat{x}$, respectively. The forces on the top and bottom edges of length $w$ become $\pm(I/c)wB\cos\theta\hat{z}$. The total force adds up to zero. As for the torque, the forces along $\hat{z}$ act colinearly and produce no torque. The pair of forces along $\hat{x}$ are not colinear and produce a torque $\vec{\tau} = \vec{w} \times \vec{F}$, where $\vec{w}$ denotes the vector length of top edge of the loop. Then,*

$$\vec{\tau} = wF\sin\theta\,\hat{z} = (I/c)wLB\sin\theta\,\hat{z}\,.$$

*b) Taking the divergence of both sides of the equation, the left hand side vanishes $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$. But, on using the continuity equation, the right hand side is proportional to $\vec{\nabla} \cdot \vec{J} = -\partial\rho/\partial t$ which is non-zero in time dependent situations. Hence in these cases, the equation is not consistent with charge conservation. To restore consistency with charge conservation, we have to modify $\vec{J}$ on the right hand side to some $\vec{J} + \cdots$ so that the new quantity always remains divergenceless. Such a quantity is obtained by combining the continuity equation with the Maxwell equation $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ which gives, $\vec{\nabla} \cdot (\vec{J} + \frac{1}{4\pi}\partial\vec{E}/\partial t) = 0$. Hence, the modified equation becomes $\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$.*

4. Consider Maxwell's equations in a medium of constant permittivity $\epsilon$ and permeability $\mu$ in the absence of sources.

   (a) Show that the equations admit plane-wave solutions. Determine the velocity of the plane wave and explain the *index of refraction*.

   (b) Show that $\vec{E}$, $\vec{B}$ and the wave vector $\vec{k}$ are all perpendicular to each other.

   **Solution** (points: 13)
   *a) To show that Maxwell's equations in the absence of sources admit plane wave solutions, one first shows that they reduce to wave equations for $\vec{E}$ and $\vec{B}$. For this, take curls of the two equations containing $\vec{\nabla} \times \vec{E}$ and $\vec{\nabla} \times \vec{B}$ and simplify using the vector identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2\vec{E} + \vec{\nabla}(\vec{\nabla} \cdot \vec{E})$. Now use the remaining two Maxwell equations to get decoupled wave equations of $\vec{E}$ and $\vec{B}$,*

$$\left(\nabla^2 - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0\,, \qquad \left(\nabla^2 - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = 0\,.$$

*These admit plane wave solutions,*

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\,, \qquad \vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\,, \qquad \text{with} \quad k^2 = \frac{\mu\epsilon}{c^2}\omega^2\,.$$

*From the wave equations or from the solutions, one can read off the wave velocity in the medium as* $v = c/\sqrt{\mu\epsilon}$. *The index of refraction for electromagnetic waves in a medium is given by* $n = c/v = \sqrt{\mu\epsilon}$.
*b) For the above solutions,* $\vec{\nabla}\cdot\vec{E} = i\vec{k}\cdot\vec{E}$ *and* $\vec{\nabla}\times\vec{E} = i\vec{k}\times\vec{E}$ *with similar results for* $\vec{B}$. *Then the Maxwell equations* $\vec{\nabla}\cdot\vec{E} = 0$ *and* $\vec{\nabla}\cdot\vec{B} = 0$ *imply* $\vec{k}\cdot\vec{E} = 0$, $\vec{k}\cdot\vec{B} = 0$. *So* $\vec{E}$ *and* $\vec{B}$ *are perpendicular to the wave vector* $\vec{k}$. *Now substituting the solutions in the remaining two Maxwell's equations gives* $\vec{k}\times\vec{E} = \frac{\omega}{c}\vec{B}$ *and* $\vec{k}\times\vec{B} = -\frac{\omega}{c}\vec{E}$ *which proves that* $\vec{E}$ *and* $\vec{B}$ *are also perpendicular to each other.*

5. (a) Starting with Maxwell's equations in the presence of sources, introduce the potentials $\vec{A}$ and $\Phi$ and rewrite the equations in terms of the potentials in the Lorenz gauge.

   (b) Write down the solution for $\vec{A}$ in terms of the spherically symmetric retarded Green's function (the Green's function need not be derived) for a localized source with a sinusoidal time dependence, $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$. How does one characterize the Near, Intermediate and Far zones? Discuss the solution in the Near zone.

**Solution** (points: 14)
*a) Let us consider the two homogeneous equations (the ones without sources).* $\vec{\nabla}\cdot\vec{B} = 0$ *implies that one can always express the magnetic field as* $\vec{B} = \vec{\nabla}\times\vec{A}$. *Using this, the remaining homogeneous equation becomes,* $\vec{\nabla}\times(\vec{E}+\frac{1}{c}\partial\vec{A}/\partial t) = 0$, *which implies the existence of a scalar potential such that* $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\partial\vec{A}/\partial t = 0$.
*Now we substitute these into the equations with sources.* $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ *gives,*

$$\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho\,.$$

*The Maxwell equation sourced by* $\vec{J}$ *gives (on using* $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A}+\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$*),*

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = -\frac{4\pi}{c}\vec{J}.$$

*The Lorenz gauge condition* $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}(\partial\Phi/\partial t) = 0$ *decouples these two equations giving,*

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho\,, \qquad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J}\,.$$

*b) The solution of the above equation for* $\vec{A}$ *in terms of the spherically symmetric retarded Green's function is*

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x'\,\frac{\left[\vec{J}(\vec{x}',t')\right]_{ret}}{|\vec{x}-\vec{x}'|}\,,$$

*where the numerator is to be evaluated at the retarded time* $t' = t - |\vec{x} - \vec{x}'|/c$. *Hence,* $\left[\vec{J}(\vec{x}', t')\right]_{ret} = \vec{J}(\vec{x}', t' = t - |\vec{x} - \vec{x}'|/c)$, *and for the given sinusoidal current,*

$$\vec{A}(\vec{x}, t) = \frac{e^{-i\omega t}}{c} \int d^3x' \frac{\vec{J}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x} - \vec{x}'|}\,,$$

*where* $k = \omega/c$ $(= 2\pi/\lambda$, *say). There are three length scales in the problem: 1) the linear extension of the current distribution denoted by* $d$ *(then, with the origin of the coordinate system chosen within the current distribution, one has* $x' \lesssim d$*), 2) the length* $\lambda$ *which is the distance that a signal travels during one oscillation of the source (note that* $2\pi/\omega = T$ *is the time period of the oscillating source), 3) the distance to the observer denoted by* $x = |\vec{x}|$. *For a well localized source, we always assume that* $d << x, \lambda$. *Now, the space around the source may be divided into three different zones depending on the position of the observer relative to the "wavelength"* $\lambda$: *(i)* $d << x << \lambda$: *"near zone", (ii)* $d << x \sim \lambda$: *"intermediate zone" and (iii)* $d << \lambda << x$: *"far zone". In the near zone, we can make the approximation* $k|\vec{x} - \vec{x}'| \sim k|\vec{x}| << 1$ *or* $e^{ik|\vec{x}-\vec{x}'|} \sim 1$, *so that*

$$\vec{A}(\vec{x}, t) = \frac{e^{-i\omega t}}{c} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}\,.$$

*Except for the overall time modulation, this has the character of a magnetostatic field.*

6. (a) Show that Maxwell's 4 equations are contained in the two relativistic equations $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$ and $\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0$. Further, show that the two relativistic equations have the same form in all inertial reference frames.

   (b) Assume that an inertial reference frame $\tilde{S}$ is moving away from a frame $S$ with velocity $v$ in the positive $x^1$ direction. If the observer in $S$ measures fields corresponding to an electrostatic *potential* $\phi(\vec{x}) = Q/x$, where $x = \sqrt{\sum_1^3 x^i x^i}$, find the electric and magnetic *potentials* as measured by the observer in $\tilde{S}$.

**Solution** (points: 14)
*a) We start by writing the equation with the source* $J^\nu$ *separately for* $\nu = 0$ *and* $\nu = j$ *(where* $j$ *is a space index). The index* $\mu$ *is summed over, so all its values are retained,*

$$\partial_i F^{i0} = \frac{4\pi}{c} J^0\,, \qquad \partial_0 F^{0j} + \partial_i F^{ij} = \frac{4\pi}{c} J^j\,,$$

*where we have used* $F^{00} = 0$. *Now we note that* $J^0 = c\rho$, $F^{i0} = E^i$, $F^{ij} = -\epsilon^{ij}{}_k B^k$ *and* $\partial_i F^{ij} = -\epsilon^{ij}{}_k \partial_i B^k = (\vec{\nabla} \times \vec{B})^j$. *Thus, we recover the two sourced Maxwell equations (you may also avoid using* $\epsilon_{ijk}$ *and write* $F_{12}$, *etc., directly in terms of* $B_i$*),*

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho\,, \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{J}.$$

*In the expression, $P_{\mu\nu\rho} = \partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu}$, the indices are not summed over. Moreover using the antisymmetry of $F_{\mu\nu}$ one can show that $P_{\mu\nu\rho}$ is antisymmetric under the exchange of any two of its indices. So it is non-zero only when $\mu, \nu, \rho$ all take different values. Now, there are too possibilities: (1) all indices take spacial values, say, $\mu = i, \nu = j, \rho = l$. Since each index can take only 3 values, all choices are equivalent to $\mu = 1, \nu = 2, \rho = 3$. (2) One index denotes time and the two other space, say, $\mu = 0, \nu = j, \rho = k$. In case (1), writing $F_{ij} = -\epsilon_{ijk}B^k$, one gets $P_{123} = -\sum_{k=1}^{3}(\epsilon_{23k}\partial_1 B^k + \epsilon_{12k}\partial_3 B^k + \epsilon_{31k}\partial_2 B^k)$. In each term, the value of the index k cannot be the same as either of the other two indices on the $\epsilon$-tensor. Hence it has to equal the third possible value, which is the index on the derivative. Then,*

$$P_{123} = -\epsilon_{123}(\partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3) = -\vec{\nabla} \cdot \vec{B} = 0\,,$$

*using $\epsilon_{123} = 1$.*

*In case (2), $P_{0ij} = \partial_0 F_{ij} + \partial_j F_{0i} + \partial_i F_{j0} = \partial_0 F^{ij} - \partial_j F^{0i} - \partial_i F^{j0}$. Now, for $\{i, j\} = \{1, 2\}, \{1, 3\}, \{2, 3\}$, one gets*

$$P_{012} = -(\partial_0 B^3 + \partial_1 E^2 - \partial_2 E^1) = -(\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t})^3\,,$$

*and the corresponding expressions for $P_{013}$ and $P_{023}$. Hence one recovers the two sourceless Maxwell equations.*

*In a different Lorentz frame, the two expressions $\partial_\mu F^{\mu\nu} - \frac{4\pi}{c}J^\nu$ and $\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu}$ take the form*

$$\tilde{\partial}_{\tilde{\mu}} \tilde{F}^{\tilde{\mu}\tilde{\nu}} - \frac{4\pi}{c}\tilde{J}^{\tilde{\nu}} = L^{\tilde{\nu}}{}_{\nu}(\partial_\mu F^{\mu\nu} - \frac{4\pi}{c}J^\nu)\,,$$

*and*

$$\tilde{\partial}^{\tilde{\mu}}\tilde{F}^{\tilde{\nu}\tilde{\rho}} + \tilde{\partial}^{\tilde{\rho}}\tilde{F}^{\tilde{\mu}\tilde{\nu}} + \tilde{\partial}^{\tilde{\nu}}\tilde{F}^{\tilde{\rho}\tilde{\mu}} = L^{\tilde{\mu}}{}_{\mu}L^{\tilde{\nu}}{}_{\nu}L^{\tilde{\rho}}{}_{\rho}(\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu})\,,$$

*where $L^{\tilde{\mu}}{}_{\mu}$ are the components of the Lorentz transformation matrix. These expressions have exactly the same form in the two frames and when the equations in the original frame are satisfied, the corresponding equations in transformed frame also hold.*

*b) The electric potential $\phi(x)$ and magnetic potential $\vec{A}(x)$ combine into a 4-vector $A^\mu = \{A^0 = \phi, \vec{A}\}$ which under Lorentz transformations $L$ transforms as*

$$\tilde{A}^\mu(\tilde{x}) = L^\mu{}_\nu A^\nu(L^{-1}\tilde{x})\,.$$

*In our case, $\vec{A} = 0$ and the non-trivial components of $L$ are, $L^0{}_0 = L^1{}_1 = \gamma$, and $L^1{}_0 = L^0{}_1 = -\gamma\beta$. Therefore Lorentz transformation gives (suppressing the $\tilde{x}$ dependence)*

$$\tilde{\phi} = \gamma\phi\,, \qquad \tilde{A}^1 = -\gamma\beta\phi\,, \qquad \tilde{A}^2 = \tilde{A}^3 = 0\,.$$

*To complete the transformation, we have to express the $x^\mu$ dependence of $\phi$ in terms of $\tilde{x}^\mu$. For the given Lorentz transformation, $x^1 = \gamma(\tilde{x}^1 + \beta\tilde{x}^0)$, $x^2 = \tilde{x}^2$ and $x^3 = \tilde{x}^3$, so that $x^2 = \sum_1^3 x^i x^i = \gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2$. Then,*

$$\tilde{\phi}(\tilde{x}) = \gamma\frac{Q}{\sqrt{(\gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2)}}\,, \qquad \tilde{A}^1 = -\beta\tilde{\phi}(\tilde{x})\,.$$