Final Examination Paper for Electrodynamics-I

Date: *Friday, Oct 31, 2008*, Time: *09:00 - 15:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks or equivalent*

`Note: Please explain your reasoning and calculations clearly`

Questions:	1	2	3	4	5	6	Total
Marks:	13	14	13	14	13	13	80

1. (a) Show that the energy density stored in an electrostatic field is given by $w = \frac{1}{8\pi}\vec{E}\cdot\vec{E}$ (or the corresponding expression in SI units).

   (b) Consider the solutions of the Poisson equation, $\nabla^2\phi = -4\pi\rho$, in a volume V with a boundary S. Show that a solution is uniquely determined by specifying *either* Neumann *or* Dirichlet boundary conditions on S.

   Solution (points: 6+7)

   a) Electrostatic fields are created by charge distributions and the energy stored in them is the energy spent in building up the charge distribution adiabatically. This can be calculated as follows: Consider a charge distribution specified by a charge density $\rho(\vec{x})$. The resulting electrostatic potential ϕ then satisfies $\nabla^2\phi(\vec{x}) = -4\pi\rho(\vec{x})$. Let us add a small extra amount of charge to change the distribution to $\rho + \delta\rho$. The electrostatic energy of the new charges in the old potential is $\delta W = \int d^3x\delta\rho(\vec{x})\phi(\vec{x})$ and this is the work required to change the charge density infinitesimally. To compute the total work, one needs to integrate this out from $\rho = 0$ to some final ρ. One way of doing this is to realize that $\delta\rho$ is associated with a change $\delta\phi$ in the potential satisfying, $\nabla^2\delta\phi(\vec{x}) = -4\pi\delta\rho(\vec{x})$. Hence, δW can be written as (on dropping a surface term and using $\vec{E} = -\vec{\nabla}\phi$),

$$\begin{aligned}\delta W &= -\frac{1}{4\pi}\int d^3x(\nabla^2\delta\phi)\phi = \frac{1}{4\pi}\int d^3x(\vec{\nabla}\delta\phi)\cdot\vec{\nabla}\phi \\ &= \frac{1}{4\pi}\int d^3x(\delta\vec{E})\cdot\vec{E} = \frac{1}{8\pi}\int d^3x\delta(\vec{E}\cdot\vec{E})\end{aligned}$$

   This clearly is a variation $W = \frac{1}{8\pi}\int d^3x(\vec{E}\cdot\vec{E})$ as the charge density is varied. From here the energy density can be read off as $w = \frac{1}{8\pi}\vec{E}\cdot\vec{E}$.

   b) The strategy is to start with two possibly different solutions ϕ_1 and ϕ_2 of the Poisson equation satifying the same boundary conditions (either Neumann or Dirichlet). Define $U = \phi_2 - \phi_1$ which then satisfies the Laplace equation. Green's first identity then leads to $U = 0$ (Dirichlet) or $U = const$ (Neumann) from which the uniqueness of the solution follows (for details of the proof, see section 1.9 of Jackson's book, or the class notes).

2. Consider an external electric field given by $E_i = C_i + D_{ij}x^j$ in a region of space free of charges and currents.

(a) Show that the matrix D_{ij} is traceless ($\sum_i D_{ii} = 0$) and symmetric ($D_{ij} = D_{ji}$). What is the external potential $\Phi_{ext}(\vec{x})$ corresponding to this electric field (ignore the undetermined constant piece)?

(b) In this external field place a *conducting* sphere of radius R centred at $\vec{x} = 0$ and carrying zero net charge. Suppose the polarisation of the sphere in the external field is described by a dipole moment p_i and a quadrupole moment Q_{ij}. Write the expression for the induced potential $\Phi_{in}(\vec{x})$ for $|\vec{x}| \geq R$ generated by the multipole moments in terms of p_i and Q_{ij}. What is the total potential $\Phi_{ext} + \Phi_{in}$ inside the sphere ($|\vec{x}| \leq R$)?

(c) Determine p_i and Q_{ij} in terms of C_i, D_{ij} and R and find the total potential $\Phi_{ext} + \Phi_{in}$ outside the sphere ($|\vec{x}| \geq R$).

(d) Compute the induced surface charge density on the sphere (Hint: If $\mathbf{e}_i$ denote the basis vectors in Cartesian coordinates, then $\vec{x} = x^i \mathbf{e}_i$ and for the radial unit vector, $\hat{x} = \vec{x}/|\vec{x}| = \hat{x}^i \mathbf{e}_i$. In spherical coordinates one can write, $x^i = x\hat{x}^i$ where $\hat{x}^3 = \cos\theta$, $\hat{x}^1 = \sin\theta\cos\phi$, $\hat{x}^2 = \sin\theta\sin\phi$. Hence, they do not vary with radial distance x).

Solution (points: 4+3+4+3)
a) The electric field satisfies $\vec{\nabla}\cdot\vec{E} = \sum_i \partial_i E^i = 0$ implying $\sum_i D_{ii} = 0$ and $(\vec{\nabla}\times\vec{E})_i = \sum_{jk} \epsilon_i{}^{jk}\partial_j E_k = 0$ implying $\sum_{jk}\epsilon_i{}^{jk} D_{jk} = 0$ or $D_{jk} = D_{kj}$. Therefore, the matrix D is traceless and symmetric. The corresponding potential, consistent with $\vec{E} = -\nabla\Phi_{ext}$, is

$$\Phi_{ext} = -\sum_i C_i x^i - \frac{1}{2}\sum_{ij} D_{ij}x^i x^j$$

(as stated in the problem, we have set the constant part of Φ_{ext} equal to zero)
b) For $|\vec{x}| \geq R$, the induced potential due to the polarized sphere is the same as that due a dipole of moment $\vec{p}$ and a quadrupole of moment matrix Q_{ij} placed at the origin,

$$\Phi_{in} = \frac{\vec{p}\cdot\vec{x}}{x^3} + \frac{1}{2}\frac{Q_{ij}x^i x^j}{x^5}$$

The total potential $\Phi_{ext} + \Phi_{in}$ inside the sphere is zero (up to a constant).
c) The sphere being conducting, the total potential $\Phi_{in} + \Phi_{ext}$ on its surface must vanish,

$$\left(\frac{p_i x^i}{R^3} + \frac{1}{2}\frac{Q_{ij}x^i x^j}{R^5} - C_i x^i - \frac{1}{2}D_{ij}x^i x^j\right)\Big|_{|\vec{x}|=R} = 0$$

Since the x^i vary on the surface, comparing terms with the same tensor structure, one gets $p_i = R^3 C_i$ and $Q_{ij} = R^5 D_{ij}$ (note that in general the total potential on the surface is a constant, not necessarily zero. However, the x^i dependence of $\Phi_{in} + \Phi_{ext}$ on the surface then shows that this constant is zero at long as we drop the constant part of Φ_{ext} as we are told to do). The total potential for $|\vec{x}| \geq R$ is then,

$$\Phi = \Phi_{in} + \Phi_{ext} = C_i x^i\left(\frac{R^3}{x^3} - 1\right) + \frac{1}{2}D_{ij}x^i x^j\left(\frac{R^5}{x^5} - 1\right)$$

d) Using the notation described in the question, one can write the total potential ϕ in spherical polar coordinates as

$$\Phi = C_i \hat{x}^i \left(\frac{R^3}{x^2} - x \right) + \frac{1}{2} D_{ij} \hat{x}^i \hat{x}^j \left(\frac{R^5}{x^3} - x^2 \right)$$

where $\hat{x}^i$ are independent of $x = |\vec{x}|$, depending only on the angular variables. The surface charge density is given by $(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\sigma$ where $\hat{n}$ is the unit normal to the surface of the sphere. In this case, $\vec{E}_1 = 0$ and $\vec{E}_2 \cdot \hat{n} = -\partial\Phi/\partial x|_{x=R}$. Therefore,

$$\sigma = -\frac{1}{4\pi} \frac{\partial \Phi}{\partial x}\bigg|_{x=R} = \frac{1}{4\pi} \left(3C_i \hat{x}^i + \frac{5}{2} R D_{ij} \hat{x}^i \hat{x}^j \right)$$

3. (a) Starting from the Biot-Savart law,

$$\vec{B}(\vec{x}) = \frac{1}{c} \int d^3x' \, \vec{J}(\vec{x}\,') \times \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3}$$

derive the two Maxwell equations involving $\vec{\nabla} \cdot \vec{B}$ and $\vec{\nabla} \times \vec{B}$ for *time dependent* situations.

(b) The magnetic moment of current distribution is given by

$$\vec{m} = \frac{1}{2c} \int d^3x' [\vec{x}\,' \times \vec{J}(\vec{x}\,')]$$

Derive the expression for $\vec{m}$ for a thin wire carrying current I. If the thin wire is a circular loop of radius a, using your formula, find the potential energy of the loop in a uniform magnetic field $\vec{B}$

Solution (points: 7+6)
a) We need the following relations: $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$ and

$$\vec{\nabla} \frac{1}{|\vec{x} - \vec{x}\,'|} = -\frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3} = -\vec{\nabla}' \frac{1}{|\vec{x} - \vec{x}\,'|}, \quad \nabla^2 \frac{1}{|\vec{x} - \vec{x}\,'|} = -4\pi \delta^3(\vec{x} - \vec{x}\,'),$$

Note that $\vec{\nabla}$ involves differentiations with respect to $\vec{x}$ while $\vec{\nabla}'$ involves differentiations with respect to $\vec{x}\,'$. This difference should be kept in mind. Using the above, the Biot-Savart law can be rewritten as $\vec{B}(\vec{x}) = \frac{1}{c} \vec{\nabla} \times \left(\int d^3x' \vec{J}(\vec{x}\,')/|\vec{x} - \vec{x}\,'| \right)$ from which it immediately follows that $\vec{\nabla} \cdot \vec{B} = 0$ (by the vector identity $\vec{\nabla} \cdot \vec{\nabla} \times (\ ) = 0$). Furthermore,

$$\begin{aligned}
\vec{\nabla} \times \vec{B}(\vec{x}) &= \frac{1}{c} \vec{\nabla} \times \left(\vec{\nabla} \times \int d^3x' \vec{J}(\vec{x}\,')/|\vec{x} - \vec{x}\,'| \right) \\
&= \frac{1}{c} \vec{\nabla} \left(\int d^3x' \vec{J}(\vec{x}\,') \cdot \vec{\nabla} \frac{1}{|\vec{x} - \vec{x}\,'|} \right) - \frac{1}{c} \int d^3x' \vec{J}(\vec{x}\,') \nabla^2 \frac{1}{|\vec{x} - \vec{x}\,'|}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{c}\vec{\nabla}\left(\int d^3x' \vec{J}(\vec{x}\,') \cdot \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}\,'|}\right) + \frac{4\pi}{c}\vec{J}(\vec{x}) \\
&= -\frac{1}{c}\vec{\nabla}\int d^3x' \left(\vec{\nabla}' \cdot \frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} - \frac{\vec{\nabla}' \cdot \vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\right) + \frac{4\pi}{c}\vec{J}(\vec{x}) \\
&= -\frac{1}{c}\vec{\nabla}\oint d\vec{S}' \cdot \frac{\vec{J}(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\int d^3x' \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|} + \frac{4\pi}{c}\vec{J}(\vec{x})
\end{aligned}$$

The first integral is a surface term and vanishes. The second integral becomes,

$$-\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\Phi(\vec{x}) = \frac{1}{c}\frac{\partial}{\partial t}\vec{E}$$

and hence $\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{J}$.
b) If $\vec{dl}$ *is a small displacement along the thin wire, then,* $\int_V d^3x' = \int_l \int_S \vec{dl} \cdot d\vec{S}$, *where* S *is the cross section area at a point* l. *Furthermore, at the same point,* $\vec{J}$ *and* $\vec{dl}$ *are parellel. Therefore,*

$$\vec{m} = \frac{1}{2c}\int_l\int_S \vec{dl} \cdot d\vec{S}[\vec{x}\,' \times \vec{J}(\vec{x}\,')] = \frac{1}{2c}\int_l\int_S \vec{J} \cdot d\vec{S}[\vec{x}\,' \times \vec{dl}]$$

For a thin wire, at any point l *along its length, the variation of* $\vec{x}\,'$ *over its cross section is very small and hence it can be replaced by some* $\vec{x}$ *which is the average of* $\vec{x}\,'$ *over the cross section. Now, the surface integral does not affect* $\vec{x}$ *and gives,* $\int_S J \cdot dS = I$, *leading to,*

$$\vec{m} = \frac{I}{2c}\int_l \vec{x} \times \vec{dl}$$

For a circular loop of radius a, *choosing the origin at the center of the loop for convenience,* $\vec{x} \times \vec{dl} = a\sin(\pi/2)\hat{n}\,dl$, *where* $\hat{n}$ *is a unit vector normal to the loop area and its direction given by the right-hand-rule relative to the direction of the current* I. *Then,* $\vec{m} = (I/2c)2\pi a^2\hat{n}$ *and the potential enery of the loop is given by* $-\vec{m} \cdot \vec{B} = -\frac{\pi a^2 I}{c}\hat{n} \cdot \vec{B}$.

4. A straight piece of cylindrical wire of length L and radius a has a resistance R and carries current I.

   (a) Find the electric and magnetic fields on the surface of the wire and indicate their directions.

   (b) Evaluate the energy carried into the wire by the above electric and magnetic fields.

   (c) What happens to this energy in the steady state? Verify your answer using the Poynting theorem in the thin wire approximation.

   Solution (points: 5+5+4)
   a)The electric field on the surface is given by $E = \Phi/L$ *where the constant potential difference* Φ *is given by Ohm's law,* $\Phi = IR$. *Hence,* $E = IR/L$. *The direction*

of $\vec{E}$ is parallel to the current and hence to the wire. The magnetic field on the surface is given by Ampere's law as $B = 2I/ca$ (This is obtained by integrating $\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{J}$ over a cross section of the wire and using the cylindrical symmetry of the problem). The direction of $\vec{B}$ is given by the "right-hand-rule" which makes it perpendicular to both $\vec{E}$ and the radius vector of the cylindrical wire. Hence the direction of $\vec{B}$ is along the angular direction of the cylinder.

b) The energy carried into the wire by electric and magnetic fields is the surface integral of the Poynting vector over the surface of the wire. The Poynting vector is $\vec{S} = (c/4\pi)\vec{E} \times \vec{B}$. Since $\vec{E}$ is perpendicular to $\vec{B}$, we have, for the magnitude of the Poynting vector, $S = I^2R/(2\pi aL)$. $\vec{S}$ is directed radially inward, $\vec{S} = -\hat{r}\,S$. The flux $\int \vec{ds} \cdot \vec{S}$ evaluated over the surface of a segment of length L of the cylindrical wire receives contributions only from the curved side-area of the cylinder (of area $2\pi aL$) and not from the top and bottom caps (since then $\vec{S}$ is perpendicular to $\vec{ds}$). Hence the total flux is

$$\int \vec{ds} \cdot \vec{S} = -I^2R$$

The sign is due to fact that $\vec{S}$ is directed radially inward while, for the cylinder, $\vec{ds}$ is directed radially outward. Physically, the negative sign signifies that energy enters into the volume under consideration, rather than leave it.

c) Since in this problem the electric and magnetic fields are constant, the Poynting theorem reduces to

$$-\int_V d^3x \vec{E} \cdot \vec{J} = \int_{\partial V} \vec{ds} \cdot \vec{S}$$

The left hand side is recognized as the expression for the energy injected into the current distribution by the electric and magnetic fields. Thus the energy carried into the wire by the Poynting vector is fully converted into the kinetic energy of the charge carriers. Since the current is constant, the system is in steady state and the extra kinetic energy acquired by the charges is dissipated into heat as a result of collisions within the resistive medium. The left hand side can be computed in the thin wire approximation. For our wire, $d^3x = \vec{dl} \cdot \vec{ds}$ where $\vec{dl}$ is along the length of the wire and the $\vec{ds}$ integration is over the cross sectional area of the wire. In the thin wire approximation, $\vec{J}$ is parallel to $\vec{dl}$, so that $(\vec{dl} \cdot \vec{ds})(\vec{J} \cdot \vec{E}) = (\vec{J} \cdot \vec{ds})(\vec{dl} \cdot \vec{E})$. Moreover, $\vec{E}$ can be taken to be constant over the cross section of the thin wire. Then

$$-\int_V d^3x \vec{E} \cdot \vec{J} \approx -(\int \vec{J} \cdot \vec{ds})(\int \vec{dl} \cdot \vec{E}) = -I\Phi = -I^2R$$

which verifies the result of part b) of the question.

5. (a) Write the expressions for the spherically symmetric advanced and retarded Greens functions and describe their physical significance.

   (b) The electric field satisfies the wave equation,

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\vec{E} = 4\pi(\frac{1}{c^2}\frac{\partial \vec{J}}{\partial t} + \vec{\nabla}\rho)$$

Write down the solution in terms of the spherically symmetric *retarded Green function* and work out Jefimenko's generalization of the Coulomb law.

Solution (points: 6+7)
a) Corresponding to a wave equation $(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\psi = -4\pi f(\vec{x}t)$*, the equation for the Greens function is*

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G = -4\pi\delta^3(\vec{x}-\vec{x}')\delta(t-t')$$

and the spherically symmetric advanced and retarded solutions are given by

$$\begin{aligned} G^+(\vec{x},t;\vec{x}',t') &= \frac{1}{R}\delta(t'-t+R/c) \qquad (retarded) \\ G^-(\vec{x},t;\vec{x}',t') &= \frac{1}{R}\delta(t'-t-R/c) \qquad (advanced) \end{aligned}$$

where, $R = |\vec{x}-\vec{x}\,'|$*. The formal solution for* ψ *in terms of* G *is,*

$$\psi(\vec{x},t) = \psi_0(\vec{x},t) + \int d^3x' \int dt' G(\vec{x},t;\vec{x}\,',t')f(\vec{x}\,',t')$$

where $\psi_0(\vec{x},t)$ *is the solution to the homogeneous equation* $(f=0)$ *and* G *is either* G^+ *or* G^-*. Clearly,* $G^+ \neq 0$ *only when* $t = t'+R/c$*. Let us interpret* t' *as the time when a signal is emitted by the source* $f(\vec{x}\,',t')$ *at point* $\vec{x}\,'$ *(signal means a variation of the source* f*). Then the above equations imply that this signal will reach a point* $\vec{x}$ *in space only at a later time* $t = t'+R/c$*, meaning that the signal requires a finite travel time* R/c*. For the advance function,* $G^- \neq 0$ *when* $t = t'-R/c$*. If we retain the same interpretation for* t' *and* $\vec{x}\,'$ *as above, then the solution with* G^- *describes a situation where a signal emitted by the source at time* t' *reached the observer at point* $\vec{x}$ *at an earlier time* $t'-R/c$*. In other words, naively, the advanced function propagates a signal backwards in time.*
b) Comparing the wave equation for $\vec{E}$ *with the equations in part a), one can write the solution at the retarded time* t *as*

$$\vec{E}(\vec{x},t) = -\frac{1}{c^2}\int d^3x' \frac{\left[\frac{\partial \vec{J}(\vec{x}',t')}{\partial t'} + \vec{\nabla}'\rho(\vec{x}',t')\right]_{t'=t-\frac{\vec{x}-\vec{x}'}{c}}}{|\vec{x}-\vec{x}'|}$$

To express $[\vec{\nabla}'\rho]_{ret}$ *in term of* $\vec{\nabla}'[\rho]_{ret}$*, note that,*

$$\frac{\partial[\rho]_{ret}}{\partial x'^i} = \frac{\partial\rho(x',t'=t-\frac{|\vec{x}-\vec{x}'|}{c})}{\partial x'^i} = \left[\frac{\partial\rho(x',t')}{\partial x'^i}\right]_{ret} + \left[\frac{\partial\rho(x',t')}{\partial t'}\right]_{ret}\frac{\partial(t-|\vec{x}-\vec{x}'|/c)}{\partial x'^i}$$

where, $\frac{\partial(t-|\vec{x}-\vec{x}'|/c)}{\partial x'^i} = -\frac{1}{c}\frac{\partial|\vec{x}-\vec{x}'|}{\partial x'^i} = \frac{1}{c}\frac{\partial|\vec{x}-\vec{x}'|}{\partial x^i}$*. In vector notation,*

$$\vec{\nabla}'[\rho(x',t')]_{ret} = \left[\vec{\nabla}'\rho(x',t')\right]_{ret} + \frac{1}{c}\left[\frac{\partial\rho(x',t')}{\partial t'}\right]_{ret}\vec{\nabla}R$$

We can use this to substitute for $[\vec{\nabla}'\rho]_{ret}$ in the expression for $\vec{E}$ in terms of $\vec{\nabla}'[\rho]_{ret}$. Now the term involving $\int d^3x'\vec{\nabla}'[\rho]_{ret}/R$ can be integrated by parts to get Jefimenko's generalization of Coulomb's law,

$$\vec{E}(\vec{x}',t') = \int d^3x' \left(\frac{[\rho(\vec{x}',t')]_{ret}}{R^2}\hat{R} + \left[\frac{\partial\rho(\vec{x}',t')}{\partial t'}\right]_{ret}\frac{\hat{R}}{Rc} - \left[\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret}\frac{1}{Rc^2}\right)$$

where $\hat{R} = \vec{\nabla}R$

6. (a) Show that Maxwell's 4 equations are contained in the two relativistic equations $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu$ and $\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0$. Further, show that the two relativistic equations have the same form in all inertial reference frames.

   (b) Assume that an inertial reference frame $\tilde{S}$ is moving away from a frame S with velocity v in the positive x^1 direction. If the observer in S measures fields corresponding to an electrostatic potential $\phi(\vec{x}) = Q/x$, where $x = \sqrt{\sum_1^3 x^i x^i}$, find the electric and magnetic potentials as measured by the observer in $\tilde{S}$. Discuss the non-relativistic limit of your result.

Solution (points: 4+4+5)

a) We start by writing the equation with the source J^ν seperately for $\nu = 0$ and $\nu = j$ (where j is a space index). The index μ is summed over so all its values are retained,

$$\partial_i F^{i0} = \frac{4\pi}{c}J^0 \qquad \partial_0 F^{0j} + \partial_i F^{ij} = \frac{4\pi}{c}J^j$$

where we have used $F^{00} = 0$. Now we note that $J^0 = c\rho$, $F^{i0} = E^i$, $F^{ij} = -\epsilon^{ij}{}_k B^k$ and $\partial_i F^{ij} = -\epsilon^{ij}{}_k \partial_i B^k = (\vec{\nabla}\times\vec{B})^j$. Thus, we recover the two sourced Maxwell equations (you may also avoid using ϵ_{ijk} and write F_{12}, etc., directly in terms of B_i),

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho\,, \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{J}$$

In the expression, $P_{\mu\nu\rho} = \partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu}$, the indices are not summed over. Moreover using the antisymmetry of $F_{\mu\nu}$ one can show that $P_{\mu\nu\rho}$ is antisymmetric under the exchange of any two of its indices. So it is non-zero only when μ,ν,ρ all take different values. Now, there are too possibilities: (1) all indices take spacial values, say, $\mu = i, \nu = j, \rho = l$. Since each index can take only 3 values, all choices are equivalent to $\mu = 1, \nu = 2, \rho = 3$. (2) One index denotes time and the two other space, say, $\mu = 0, \nu = j, \rho = k$. In case (1), writing $F_{ij} = -\epsilon_{ijk}B^k$, one gets $P_{123} = -\sum_{k=1}^3(\epsilon_{23k}\partial_1 B^k + \epsilon_{12k}\partial_3 B^k + \epsilon_{31k}\partial_2 B^k)$. In each term, the value of the index k cannot be the same as either of the other two indices on the ϵ-tensor. Hence it has to equal the third possible value, which is the index on the derivative. Then,

$$P_{123} = -\epsilon_{123}(\partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3) = -\vec{\nabla}\cdot\vec{B} = 0$$

using $\epsilon_{123} = 1$.

In case (2), $P_{0ij} = \partial_0 F_{ij} + \partial_j F_{0i} + \partial_i F_{j0} = \partial_0 F^{ij} - \partial_j F^{0i} - \partial_i F^{j0}$. *Now, for* $\{i,j\} = \{1,2\},\{1,3\},\{2,3\}$, *one gets*

$$P_{012} = -(\partial_0 B^3 + \partial_1 E^2 - \partial_2 E^1) = -(\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t})^3$$

and the corresponding expressions for P_{013} *and* P_{023}. *Hence one recovers the two sourceless Maxwell equations.*

In a different Lorentz frame, the two expressions $\partial_\mu F^{\mu\nu} - \frac{4\pi}{c}J^\nu$ *and* $\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu}$ *take the form*

$$\tilde{\partial}_{\tilde{\mu}} \tilde{F}^{\tilde{\mu}\tilde{\nu}} - \frac{4\pi}{c}\tilde{J}^{\tilde{\nu}} = L^{\tilde{\nu}}{}_{\nu}(\partial_\mu F^{\mu\nu} - \frac{4\pi}{c}J^\nu)$$

and

$$\tilde{\partial}^{\tilde{\mu}} \tilde{F}^{\tilde{\nu}\tilde{\rho}} + \tilde{\partial}^{\tilde{\rho}} \tilde{F}^{\tilde{\mu}\tilde{\nu}} + \tilde{\partial}^{\tilde{\nu}} \tilde{F}^{\tilde{\rho}\tilde{\mu}} = L^{\tilde{\mu}}{}_{\mu} L^{\tilde{\nu}}{}_{\nu} L^{\tilde{\rho}}{}_{\rho}(\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu})$$

where $L^{\tilde{\mu}}{}_{\mu}$ *are the components of the Lorentz transformation matrix. These expressions have exactly the same form in the two frames and when the equations in the original frame are stisfied, the corresponding equations in transformed frame also hold.*

b) The electric potential $\phi(x)$ *and magnetic potential* $\vec{A}(x)$ *combine into a 4-vector* $A^\mu = \{A^0 = \phi, \vec{A}\}$ *which under Lorentz transformations* L *transforms as*

$$\tilde{A}^\mu(\tilde{x}) = L^\mu_{\ \nu} A^\nu(L^{-1}\tilde{x})$$

In our case, $\vec{A} = 0$ *and the non-trivial components of* L *are,* $L^0_{\ 0} = L^1_{\ 1} = \gamma$, *and* $L^1_{\ 0} = L^0_{\ 1} = -\gamma\beta$. *Therefore Lorentz transformation gives (suppressing the* $\tilde{x}$ *dependence)*

$$\tilde{\phi} = \gamma\phi\,, \qquad \tilde{A}^1 = -\gamma\beta\phi\,, \qquad \tilde{A}^2 = \tilde{A}^3 = 0$$

To complete the transformation, we have to express the x^μ *dependence of* ϕ *in terms of* $\tilde{x}^\mu$. *For the given Lorentz transformation,* $x^1 = \gamma(\tilde{x}^1 + \beta\tilde{x}^0)$, $x^2 = \tilde{x}^2$ *and* $x^3 = \tilde{x}^3$, *so that* $x^2 = \sum_1^3 x^i x^i = \gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2$. *Then,*

$$\tilde{\phi}(\tilde{x}) = \gamma\,\frac{Q}{\sqrt{(\gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2)}}\,, \qquad \tilde{A}^1 = -\beta\tilde{\phi}(\tilde{x})$$

In the non-relativistic limit, $\beta = v/c \to 0$ *and* $\gamma \to 1$ *so that,*

$$\tilde{\phi}(\tilde{x}) = \frac{Q}{\sqrt{(\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2}}\,, \qquad \tilde{A}^1 \to 0$$