Final Examination Paper for Electrodynamics-I

Date: *Thursday, Jan 03, 2008,* Time: *09:00 - 15:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks or equivalent*

`Note: Please explain your reasoning and calculations clearly`

Questions:	1	2	3	4	5	6	Total
Marks:	13	14	14	13	13	13	80

1. Consider a grounded conducting sphere of radius a centred at the origin of the coordinate system. Place a point charge q at position $\vec{y}$ outside the sphere.

   (a) Construct the *image problem* by finding the value q' and the position $\vec{y}\,'$ of the image charge inside the sphere. Evaluate the potential $\phi(\vec{x})$ outside the sphere.

   (b) Write the expression for the force that acts on the charge q due to the charge that the conducting sphere has soaked up from the ground.

   (c) Compute the total charge transfered to the conducting sphere from the ground.

   Solution (points: 6+4+3)
   a) The potential due to q and q' at $\vec{x}$ outside the sphere is

$$\phi(\vec{x}) = \frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{|\vec{x}-\vec{y}\,'|}$$

   Then, $\phi(|\vec{x}|=a)=0$ gives $q'=-aq/y$ and $y'=a^2/y$ (work out the details). The potential at any point $\vec{x}$ can now be easily written using the image problem (with $\hat{x}$ and $\hat{y}$ unit vectors along $\vec{x}$ and $\vec{y}$, respectively)

$$\phi(\vec{x}) = \frac{q}{|x\hat{x}-y\hat{y}|} - \frac{aq}{y}\frac{1}{|x\hat{x}-\frac{a^2}{y}\hat{y}|}$$

   b) The force on q due to the sphere is the same as that due to q'. Hence,

$$\vec{F} = \frac{qq'}{|\vec{y}-\vec{y}'|^2}\hat{y} = -\frac{q^2(a/y)}{(y-a^2/y)^2}\hat{y}$$

   c) Consider a surface S that fully encloses the sphere and lies very close to its surface. The total charge on the sphere is then given by the Gauss law as $(1/4\pi)\int_S \vec{ds}\cdot\vec{E}$, where $\vec{E}$ is the electric field on the surface S. The field $\vec{E}$ outside the sphere is the same as that in the image problem and therefore the integral gives the total charge of the sphere as q'.

2. Consider an external electric field given by $E_i = C_i + D_{ij}x^j$ in a region of space free of charges and currents.

(a) Show that the matrix D_{ij} is traceless ($\sum_i D_{ii} = 0$) and symmetric ($D_{ij} = D_{ji}$). What is the external potential $\Phi_{ext}(\vec{x})$ corresponding to this electric field (ignore the undetermined constant piece)?

(b) In this external field place a *conducting* sphere of radius R centred at $\vec{x} = 0$ and carrying zero net charge. Suppose the polarisation of the sphere in the external field is described by a dipole moment p_i and a quadrupole moment Q_{ij}. Write the expression for the induced potential $\Phi_{in}(\vec{x})$ for $|\vec{x}| \geq R$ generated by the multipole moments in terms of p_i and Q_{ij}. What is the total potential $\Phi_{ext} + \Phi_{in}$ inside the sphere ($|\vec{x}| \leq R$)?

(c) Determine p_i and Q_{ij} in terms of C_i, D_{ij} and R and find the total potential $\Phi_{ext} + \Phi_{in}$ outside the sphere ($|\vec{x}| \geq R$).

(d) Compute the induced surface charge density on the sphere (Hint: In spherical coordinates one can write, $x^i = x\hat{x}^i$ where $\hat{x}^i$ are the Cartesian components of the radial unit vector $\hat{x}$, *e.g.*, $\hat{x}^3 = \cos\theta$, $\hat{x}^1 = \sin\theta\cos\phi$, $\hat{x}^2 = \sin\theta\sin\phi$. Hence, they do not vary with radial distance x).

Solution (points: 4+3+4+3)
a) The electric field satisfies $\vec{\nabla}\cdot\vec{E} = \sum_i \partial_i E^i = 0$ *implying* $\sum_i D_{ii} = 0$ *and* $(\vec{\nabla}\times\vec{E})_i = \sum_{jk} \epsilon_i{}^{jk}\partial_j E_k = 0$ *implying* $\sum_{jk}\epsilon_i{}^{jk} D_{jk} = 0$ *or* $D_{jk} = D_{kj}$. *Therefore, the matrix* D *is traceless and symmetric. The corresponding potential, consistent with* $\vec{E} = -\nabla\Phi_{ext}$, *is*

$$\Phi_{ext} = -\sum_i C_i x^i - \frac{1}{2}\sum_{ij} D_{ij}x^i x^j$$

(as stated in the problem, we have set the constant part of Φ_{ext} *equal to zero)*
b) For $|\vec{x}| \geq R$, *the induced potential due to the polarized sphere is the same as that due a dipole of moment* $\vec{p}$ *and a quadrupole of moment matrix* Q_{ij} *placed at the origin,*

$$\Phi_{in} = \frac{\vec{p}\cdot\vec{x}}{x^3} + \frac{1}{2}\frac{Q_{ij}x^i x^j}{x^5}$$

The total potential $\Phi_{ext} + \Phi_{in}$ *inside the sphere is zero (up to a constant).*
c) The sphere being conducting, the total potential $\Phi_{in} + \Phi_{ext}$ *on its surface must vanish,*

$$\left(\frac{p_i x^i}{R^3} + \frac{1}{2}\frac{Q_{ij}x^i x^j}{R^5} - C_i x^i - \frac{1}{2}D_{ij}x^i x^j\right)\Bigg|_{|\vec{x}|=R} = 0$$

Since the x^i *vary on the surface, comparing terms with the same tensor structure, one gets* $p_i = R^3 C_i$ *and* $Q_{ij} = R^5 D_{ij}$ *(note that in general the total potential on the surface is a constant, not necessarily zero. However, the* x^i *dependence of* $\Phi_{in} + \Phi_{ext}$ *on the surface then shows that this constant is zero at long as we drop the constant part of* Φ_{ext} *as we are told to do). The total potential for* $|\vec{x}| \geq R$ *is then,*

$$\Phi = \Phi_{in} + \Phi_{ext} = C_i x^i\left(\frac{R^3}{x^3} - 1\right) + \frac{1}{2}D_{ij}x^i x^j\left(\frac{R^5}{x^5} - 1\right)$$

d) Using the notation described in the question, one can write the total potential ϕ in spherical polar coordinates as

$$\Phi = C_i \hat{x}^i \left(\frac{R^3}{x^2} - x \right) + \frac{1}{2} D_{ij} \hat{x}^i \hat{x}^j \left(\frac{R^5}{x^3} - x^2 \right)$$

where $\hat{x}^i$ are independent of $x = |\vec{x}|$, depending only on the angular variables. The surface charge density is given by $(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\sigma$ where $\hat{n}$ is the unit normal to the surface of the sphere. In this case, $\vec{E}_1 = 0$ and $\vec{E}_2 \cdot \hat{n} = -\partial\Phi/\partial x|_{x=R}$. Therefore,

$$\sigma = -\frac{1}{4\pi} \frac{\partial \Phi}{\partial x}\bigg|_{x=R} = \frac{1}{4\pi} \left(3C_i \hat{x}^i + \frac{5}{2} R D_{ij} \hat{x}^i \hat{x}^j \right)$$

3. (a) Using the expansion

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)$$

develop the multipole expansion of the potential $\Phi(\vec{x})$ due to a localized charge distribution $\rho(\vec{x}')$ in terms of the multipole moments q_{lm} of ρ. Discuss how and under what conditions this expansion can be used to simplify a problem.

(b) Show that for a spherically symmetric charge distribution, all multipole moments beyond the monopole moment vanish.

(c) Show that if the charge distribution has axial symmetry (that is, it is invariant under rotations about the z-axis), then the only non-zero multipole moments are q_{l0}.

(d) Using the above results, for two point charges q and $-q$ placed on the z-axis at $z = a$ and $z = -a$, compute the non-vanishing component of the dipole moment (given $Y_{10} = (\sqrt{3/4\pi}) \cos\theta$).

Solution (points: 5+3+3+3)

a) The potential due to a localized charge distribution is given by

$$\Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}'|}$$

Using the expansion given in the question, it becomes,

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

This is the multipole expansion of the potential in terms of the multipole moments q_{lm} of the charge distribution given by

$$q_{lm} = \int d^3x' \rho(\vec{x}\,') r'^l Y^*_{lm}(\theta', \phi')$$

The multipole expansion allows us to parametrize the charge distribution in terms of its multipole moments. Further, the contribution of a moment q_{lm} to the potential falls off as $1/r^{l+1}$. Therefore, at large distances from a localized charge distribution, only a few non-zero multipole moments with the lowest values of l make significant contributions to Φ and are relevant. The remaining moments could be neglected. This allows us to parametrize even complicated charge distributions in terms of a few lowest l mutipole moments. The condition under which this approximation is valid is that the distance to the observation point (at which Φ is measured) is much larger as compared to the size of the charge distribution.

b) For a spherically symmetric charge distribution, $\rho(\vec{x}) \equiv \rho(r,\theta,\phi) = \rho(r)$, independent of the angular variables. Therefore we can write the multipole moments as a product of the radial and angular integrals,

$$q_{lm} = \left(\int_0^\infty r'^2 dr' \rho(r') r'^l \right) \left(\int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' Y^*_{lm}(\theta',\phi') \right)$$

Since $Y_{00}(\theta',\phi') = 1/\sqrt{4\pi}$, we can insert $Y_{00}(\theta',\phi')\sqrt{4\pi} = 1$ in the angular integration. Now, from the orthogonality property of spherical harmonics it follows that the angular integral is proportional to δ_{l0} and hence vanishes for all $l \geq 1$ ($l = 0$ being the monopole moment).

c) In the case of axial symmetry about the z-axis, ρ is independent of the azimuthal coordinate ϕ. In this case,

$$q_{lm} = (const) \left(\int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \rho(r',\theta') r'^l P_l^m(\cos\theta') \right) \left(\int_0^{2\pi} d\phi' e^{-im\phi'} \right)$$

where we have used the fact that $Y_{lm}(\theta,\phi) = (const) P_l^m(\cos\theta) e^{im\phi}$. Now, the ϕ integral gives a δ_{m0}. Thus the only non-vanishing moments in this case are q_{l0}.

d) In this case the charge density is given by $\rho = q\delta(x')\delta(y')\left(\delta(z'-a) - \delta(z'+a)\right)$. The three components of the dipole moment are q_{1m}, for $m = 1, 0, -1$. Since the problem has axial symmetry about the z-axis, the only non-vanishing component is q_{10} which is now given by (using the expressions for Y_{10}, ρ and noting that $r' = \sqrt{x'^2 + y'^2 + z'^2}$)

$$\begin{aligned} q_{10} &= \int d^3x' \rho(\vec{x}\,') r' Y^*_{10}(\theta',\phi') = \sqrt{\frac{3}{4\pi}} \int dx' \int dy' \int dz' \\ &\quad \times q\delta(x')\delta(y')\left(\delta(z'-a) - \delta(z'+a)\right) \sqrt{x'^2 + y'^2 + z'^2} \cos\theta \\ &= \sqrt{\frac{3}{4\pi}} \int dz' q \left(\delta(z'-a) - \delta(z'+a)\right) |z'| = aq\sqrt{\frac{3}{\pi}} \end{aligned} \tag{1}$$

4. (a) Consider moving charges giving rise to a current density $\vec{J}$ within a volume V in the presence of electric and magnetic fields. Show that the total power injected into the current distribution by the fields is given by $\int_V d^3x \, \vec{J} \cdot \vec{E}$.

(b) Using Maxwell's equations, derive the *Poynting theorem* [You may need the vector identity $\nabla \cdot (\vec{P} \times \vec{Q}) = (\nabla \times \vec{P}) \cdot \vec{Q} - \vec{P} \cdot (\nabla \times \vec{Q})$].

(c) Give the physical interpretation of each term in the mathematical expression for the Poynting theorem. What is the physical meaning of the Poynting theorem?

Solution (points: 5+4+4)

a) The power transferred to a point charge q on which a force $\vec{F}$ acts is the rate of change of its kinetic energy, $\frac{1}{2}mv^2$, that is, $d(\frac{1}{2}mv^2)/dt = \vec{F} \cdot \vec{v}$. Using the Lorentz force law and $\vec{v} \cdot (\vec{v} \times \vec{B}) = 0$, this becomes $q\vec{v} \cdot \vec{E}$. For charges contained in volume d^3x within a continuous charge distribution, one has $q \to \rho d^3x$. Using $\vec{J} = \rho\vec{v}$ and integrating over the volume of the current distribution, leads to the desired result.

b) To obtain the Poynting theorem, start with $\vec{J} \cdot \vec{E}$ and, using Maxwell's equations, rewrite $\vec{J}$ in terms of $\vec{E}$ and $\vec{B}$. After some manipulations, one gets,

$$\vec{J} \cdot \vec{E} + \frac{1}{8\pi} \frac{\partial}{\partial t} \left(\epsilon \vec{E} \cdot \vec{E} + \frac{1}{\mu} \vec{B} \cdot \vec{B} \right) + \frac{c}{4\pi} \nabla \cdot (\vec{E} \times \vec{H}) = 0$$

c) $\vec{J} \cdot \vec{E}$: power injected into the current distribution by the electric field/unit volume.
$\frac{1}{8\pi} \frac{\partial}{\partial t} \left(\epsilon \vec{E} \cdot \vec{E} + \frac{1}{\mu} \vec{B} \cdot \vec{B} \right)$: Rate of change of energy densities of the electric and magnetic fields.
$\frac{c}{4\pi} \nabla \cdot (\vec{E} \times \vec{H})$: Energy flux per unit time per unit volume carried by the electromagnetic fields. $\frac{c}{4\pi}(\vec{E} \times \vec{H})$ is the Poynting vector that corresponds to the energy flux per unit area per unit time across a surface as follows from the divergence theorem, $\int_V d^3x \nabla \cdot (\vec{E} \times \vec{H}) = \int_S d\vec{S} \cdot (\vec{E} \times \vec{H})$. The Poynting theorem is a statement of conservation of energy and also indicates that energy is carried by electromagnetic waves in the form of the Poynting vector.

5. (a) Consider the wave equation $(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})\psi = -4\pi f(\vec{x}, t)$. Write down the equation for the corresponding Green function G and express the formal solution for ψ in terms of G. For the retarded Green function $G(\vec{x}, t; \vec{x}', t') = \frac{1}{R}\delta(t' - t + R/c)$ (where $R = |\vec{x} - \vec{x}'|$) provide a physical interpretation for the behaviour of the solution.

(b) Maxwell's equations lead to the following wave equation for the electric field,

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})\vec{E} = 4\pi(\frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} + \vec{\nabla}\rho)$$

Write down the solution in terms of the *retarded Green function* and by re-expressing $[\vec{\nabla}'\rho]_{ret}$ in term of $\vec{\nabla}'[\rho]_{ret}$, work out Jefimenko's generalization of the Coulomb law.

Solution (points: 6+7)

a) The equation for the Greens function is

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G = -4\pi\delta^3(\vec{x} - \vec{x}')\delta(t - t')$$

The formal solution for ψ in terms of G is given by

$$\psi(\vec{x},t) = \psi_0(\vec{x},t) + \int d^3x' \int dt' G(\vec{x},t;\vec{x}\,',t') f(\vec{x}\,',t')$$

where $\psi_0(\vec{x},t)$ is the solution to the homogeneous equation ($f=0$). For the retarded Greens function $G(\vec{x},t;\vec{x}',t') = \frac{1}{R}\delta(t'-t+R/c)$, the solution becomes,

$$\psi(\vec{x},t) = \int d^3x' \frac{[f(\vec{x}',t')]_{t'=t-\frac{|\vec{x}-\vec{x}'|}{c}}}{|\vec{x}-\vec{x}'|}$$

where we have evaluated the time integral and dropped ψ_0 for simplicity. The physical interpretation of the solution is as follows: A variation of the source f at point $\vec{x}'$ and time t' affects the field ψ at a point $\vec{x}$ at a later time t provided $t = t' + \frac{|\vec{x}-\vec{x}'|}{c}$ or equivalently, $|\vec{x}-\vec{x}'| = c(t-t')$. Thus the information about the variation of f travels at speed c.

b) Comparing the wave equation for $\vec{E}$ with the equation in part a), one can write the solution as

$$\vec{E}(\vec{x},t) = -\frac{1}{c^2}\int d^3x' \frac{\left[\frac{\partial \vec{J}(\vec{x}',t')}{\partial t'} + \vec{\nabla}'\rho(\vec{x}',t')\right]_{t'=t-\frac{\vec{x}-\vec{x}'}{c}}}{|\vec{x}-\vec{x}'|}$$

To express $[\vec{\nabla}'\rho]_{ret}$ in term of $\vec{\nabla}'[\rho]_{ret}$, note that,

$$\frac{\partial[\rho]_{ret}}{\partial x'^i} = \frac{\partial\rho(x',t'=t-\frac{|\vec{x}-\vec{x}'|}{c})}{\partial x'^i} = \left[\frac{\partial\rho(x',t')}{\partial x'^i}\right]_{ret} + \left[\frac{\partial\rho(x',t')}{\partial t'}\right]_{ret} \frac{\partial(t-|\vec{x}-\vec{x}'|/c)}{\partial x'^i}$$

where, $\frac{\partial(t-|\vec{x}-\vec{x}'|/c)}{\partial x'^i} = -\frac{1}{c}\frac{\partial|\vec{x}-\vec{x}'|}{\partial x'^i} = \frac{1}{c}\frac{\partial|\vec{x}-\vec{x}'|}{\partial x^i}$. In vector notation,

$$\vec{\nabla}'[\rho(x',t')]_{ret} = \left[\vec{\nabla}'\rho(x',t')\right]_{ret} + \frac{1}{c}\left[\frac{\partial\rho(x',t')}{\partial t'}\right]_{ret}\vec{\nabla}R$$

We can use this to substitute for $[\vec{\nabla}'\rho]_{ret}$ in the expression for $\vec{E}$ in terms of $\vec{\nabla}'[\rho]_{ret}$. Now the term involving $\int d^3x'\vec{\nabla}'[\rho]_{ret}/R$ can be integrated by parts to get Jefimenko's generalization of Coulomb's law,

$$\vec{E}(\vec{x}',t') = \int d^3x' \left(\frac{[\rho(\vec{x}',t')]_{ret}}{R^2}\hat{R} + \left[\frac{\partial\rho(\vec{x}',t')}{\partial t'}\right]_{ret}\frac{\hat{R}}{Rc} - \left[\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret}\frac{1}{Rc^2}\right)$$

where $\hat{R} = \vec{\nabla}R$

6. (a) Consider the linear transformation $\tilde{x}^\mu = L^\mu_{\ \nu}x^\nu$. What are the conditions on the matrix L for this to be a Lorentz transformation? [You need not derive this]. In this case, derive an expression for L^{-1} in terms of L^T.

   (b) Show that the the continuity equation $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{J}$ follows from $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu$.

(c) Assume that an inertial reference frame $\tilde{S}$ is moving away from a frame S with velocity v in the positive x^1 direction. If the observer in S measures a static charge distribution $\rho(\vec{x}) = Qe^{-x^2/a}$, where $x^2 = \sum_1^3 x^i x^i$, find the charge and current distributions as measured by the observer in $\tilde{S}$. Discuss the non-relativistic limit of your result.

Solution (points: 4+4+5)

a) The condition on L is

$$L^T \eta L = \eta\,, \qquad \textit{where}, \qquad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(This follows from the invariance of the space-time interval, $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x^T \eta x$ under the transformation $\tilde{x} = Lx$, implying $\tilde{x}^T = x^T L^T$. Here L is the 4×4 matrix with elements $L^\mu_{\ \nu}$, with μ running over rows and ν running over the columns of the matrix. Then $x^T \eta x = \tilde{x}^T \eta \tilde{x}$ leads to the above condition.)

The defining equation for L, i.e., $L^T \eta L = \eta$ implies (on multiplying from the left by η^{-1} and from the right by L^{-1}) that, $L^{-1} = \eta^{-1} L^T \eta$

b) Differentiating $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$ with respect to x^ν gives $\partial_\nu \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} \partial_\nu J^\nu$. Now, $\partial_\nu \partial_\mu = \partial_\mu \partial_\nu$ but $F^{\mu\nu} = -F^{\nu\mu}$. Therefore $\partial_\nu \partial_\mu F^{\mu\nu} = 0$ and hence $\partial_\nu J^\nu = 0$ which is the continuity equation in the 4-vector notation, where $x^0 = ct$ and $J^0 = c\rho$.

c) The charge density $\rho(\vec{x})$ and current density $\vec{J}(\vec{x})$ combine into a 4-vector $J^\mu = \{J^0 = c\rho, \vec{J}\}$ which under Lorentz transformations L transforms as

$$\tilde{J}^\mu(\tilde{x}) = L^\mu_{\ \nu} J^\nu (L^{-1}\tilde{x})$$

In our case, $\vec{J} = 0$ and the non-trivial components of L are, $L^0_{\ 0} = L^1_{\ 1} = \gamma$, and $L^1_{\ 0} = L^0_{\ 1} = -\gamma\beta$. Therefore the Lorentz transformation gives (suppressing the $\tilde{x}$ dependence)

$$\tilde{\rho} = \gamma\rho\,, \qquad \tilde{J}^1 = -\gamma v \rho\,, \qquad \tilde{J}^2 = \tilde{J}^3 = 0$$

To complete the transformation, we have to express the x^μ dependence of ρ in terms of $\tilde{x}^\mu$. For the given Lorentz transformation, $x^1 = \gamma(\tilde{x}^1 + \beta\tilde{x}^0)$, $x^2 = \tilde{x}^2$ and $x^3 = \tilde{x}^3$, so that $x^2 = \sum_1^3 x^i x^i = \gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2$. Then,

$$\tilde{\rho}(\tilde{x}) = \gamma\, Q e^{-(\gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2)/a},, \qquad \tilde{J}^1 = -v\tilde{\rho}(\tilde{x})$$

In the non-relativistic limit, $\beta = v/c \to 0$ and $\gamma \to 1$ so that,

$$\tilde{\rho}(\tilde{x}) = Q e^{-((\tilde{x}^1 + v\tilde{t})^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2)/a},, \qquad \tilde{J}^1 = -v\tilde{\rho}(\tilde{x})$$

which is the expected result from Galilean transformations.