Final Examination Paper for Electrodynamics-I

Date: *Friday, Nov 02, 2007,* Time: *09:00 - 15:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks or equivalent*

`Note: Please explain your reasoning and calculations clearly`

Questions:	1	2	3	4	5	6	Total
Marks:	13	13	14	13	13	14	80

1. (a) Consider an electric field $\vec{E} = \hat{i}x + \hat{j}z + \hat{k}(f(x,y) + z^2)$. Determine $f(x,y)$ and compute the total charge contained in a cube specified by $0 \leq x, y, z \leq l$.

   (b) The electrostatic potential of a neutral atom can be modelled by

$$\Phi(\vec{r}) = \frac{q}{r}\, e^{-r/a}$$

   where $q = Ze$ is the atomic charge. Find the charge distribution ρ that produces this potential and show that the total charge is zero. (You may use $\vec{\nabla} r = \hat{r}$ and $\nabla^2 r = 2/r$)

Solution (points: 6+7)

a) This is an electrostatic field with $E_x = x$, $E_y = z$, $E_z = f(x,y) + z^2$ and should satisfy $\vec{\nabla} \times \vec{E} = 0$. In terms of components of $\vec{E}$ this gives $\partial E_i/\partial x^j - \partial E_j/\partial x^i = 0$ for the indices i and j taking the values x, y, z, which, in turn, leads to $\partial f/\partial x = 0$ and $\partial f/\partial y - 1 = 0$. The unknown function $f(x,y)$ is therefore given by $f = y + c$ for an arbitrary constant c. So we have, $\vec{E} = \hat{i}x + \hat{j}z + \hat{k}(y + z^2 + c)$. From this, we can compute the charge density using $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ and get $\rho = (2z+1)/4\pi$. The total charge is then given by

$$Q = \frac{1}{4\pi}\int_0^l dx \int_0^l dy \int_0^l dz(1+2z) = \frac{1}{4\pi}(l^3 + l^4)$$

(the total charge can also be computed using the Gauss law)

b) The charge distribution can be determined using $\nabla^2\Phi = -4\pi$ and is given by

$$\rho = qe^{-r/a}\left(\delta(r) - \frac{1}{4\pi}\frac{1}{ra^2}\right)$$

This clearly corresponds to a positive nuclear point charge and a negative electronic charge cloud surrounding it. The total charge is given by

$$Q = \int d^3x\rho = q - \frac{q}{a^2}\int_0^\infty e^{-r/a} r\, dr = 0$$

where we have used $\int d^3x = \int \sin\theta d\theta \int d\phi \int r^2 dr = 4\pi \int r^2 dr$ because of the spherical symmetry of the problem, along with $\int_0^\infty e^{-r/a}\, dr = a$, which on differentiating with respect to "a" gives $\int_0^\infty e^{-r/a} r\, dr = a^2$.

2. Consider the boundary between two media of dielectric constants ϵ_1 and ϵ_2 and let the electric displacement vectors on the two sides of the boundary be denoted by $\vec{D}_1$ and $\vec{D}_2$, and the polarization densities by $\vec{P}_1$ and $\vec{P}_2$, respectively. In the absence of free charges on the boundary, Maxwell equations are $\vec{\nabla} \cdot \vec{D} = 0$ and $\vec{\nabla} \times \vec{E} = 0$.

   (a) Use these equations to investigate the continuity of the normal and tangential components of $\vec{D}$ and $\vec{E}$ across the boundary.

   (b) Show that the polarization surface charge density that develops on the boundary is given by
   $$\sigma_{pol} = (\vec{P}_1 - P_2) \cdot \hat{n} \,,$$
   where $\hat{n}$ is a unit normal to the boundary.

Solution (points: 8+5)
a) In general, in steady state the fields $\vec{D}$ and $\vec{E}$ satisfy $\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f$ and $\vec{\nabla} \times \vec{E} = 0$, where ρ_f. is the density of free charges. To explore the behaviour of the normal component of $\vec{D}$, first, draw a small, so called, "Gaussian pill-box" of height h across the boundary. The top and bottom faces of the pill-box have areas ΔS each and are parallel to the boundary surface. Denote the value of the displacement field on the bottom face of the box by $\vec{D}_1$ and on the top face of the box by $\vec{D}_2$. The unit normals to these faces are $\hat{n}_1$ and $\hat{n}_2$ ($\hat{n}_2 = -\hat{n}_1 = \hat{n}$). Then, integrating $\vec{\nabla} \cdot \vec{D}$ over the pill-box volume and using the divergence theorem gives,

$$\lim_{h \to 0} \int_{pill-box} d^3x \vec{\nabla} \cdot \vec{D} = \lim_{h \to 0} \int_{\partial(pill-box)} \vec{ds} \cdot \vec{D} = (\vec{D}_2 \cdot \hat{n}_2 + \vec{D}_1 \cdot \hat{n}_1)\Delta S = 4\pi\sigma_f \Delta S$$

where σ_f is the density of free charges on the boundary and the contribution from the sides have dropped in the limit $h \to 0$. Hence we have,

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = 4\pi\sigma_f$$

In our problem, there are no free charges on the boundary and hence $(\vec{D}_2 - \vec{D}_1) \cdot \hat{n} = 0$. So the normal component of the $\vec{D}$ field is continuous across the boundary.

To investigate the behaviour of the tengential component of the field, let us now replace the pill-box by a rectangular loop that has its longer sides of length Δl parallel to the surface and its shorter sides of height h perpendicular to the surface and going through it. A unit vector along the lower side of the rectangle is $\hat{t}_1$ and one along the upper side is $\hat{t}_2$, both being parallel to the surface. Picking an orientation along the loop, one has $\hat{t}_2 = -\hat{t}_1 = \hat{t}$. Integrate $\vec{\nabla} \times \vec{E}$ over the loop area to get

$$\lim_{h \to 0} \int_{loop\ area} \vec{\nabla} \times \vec{E} \cdot \vec{ds} = \lim_{h \to 0} \int_{loop} \vec{E} \cdot \vec{dl} = (\vec{E}_2 \cdot \hat{t}_2 + \vec{E}_1 \cdot \hat{t}_1)\Delta l$$

where in the limit $h \to 0$ we have dropped the contributions from the sides of the loop. This is true for all orientations of the loop, or equivalently, for all unit tangent vectors $\hat{t}$ to the surface. Therefore we have, $(\vec{E}_2 - \vec{E}_1) \cdot \hat{t} = 0$ or equivalently,

$(\vec{E}_2 - \vec{E}_1) \times \hat{n} = 0$. *Hence the tangential component of $\vec{E}$ is continuous across the surface.*

b) Applying the pill-box construction above to $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ where $\rho = \rho_f + \rho_{nf}$ includes both free and non-free (i.e., bound) charges, one gets,

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi(\sigma_f + \sigma_{nf})$$

Now, for $\sigma_f = 0$ on the boundary, $\vec{D} \cdot \hat{n}$ is continuous across the boundary. Using this and the relation $\vec{D} = \vec{E} + 4\pi\vec{P}$, one has,

$$(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi(\vec{P}_1 - P_2) \cdot \hat{n}$$

From this we can read off the surface density of non-free charges on the boundary which are due to the polarization of the media as $\sigma_{nf} = \sigma_{pol} = (\vec{P}_1 - P_2) \cdot \hat{n}$. In short, this directly follows from the fact that $\vec{D}$ accross the surface is continuous while the discontinuity in $\vec{E} \cdot \hat{n}$ is given by the surface charge density and $\vec{D} = \vec{E} + 4\pi\vec{P}$.

3. (a) Using the expansion

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)$$

develop the multipole expansion of the potential $\Phi(\vec{x})$ due to a localized charge distribution $\rho(\vec{x}')$ in terms of the multipole moments q_{lm} of ρ. Discuss how and under what conditions this expansion can be used to simplify a problem.

(b) Show that for a spherically symmetric charge distribution, all multipole moments beyond the monopole moment vanish.

(c) Show that if the charge distribution has axial symmetry (that is, it is invariant under rotations about the z-axis), then the only non-zero multipole moments are q_{l0}.

(d) Using the above results, for two point charges q and $-q$ placed on the z-axis at $z = a$ and $z = -a$, compute the non-vanishing component of the dipole moment (given $Y_{10} = (\sqrt{3/4\pi}) \cos\theta$).

Solution (points: 5+3+3+3)

a) The potential due to a localized charge distribution is given by

$$\Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}'|}$$

Using the expansion given in the question, it becomes,

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

This is the multipole expansion of the potential in terms of the multipole moments q_{lm} of the charge distribution given by

$$q_{lm} = \int d^3x'\rho(\vec{x}\,')r'^l Y^*_{lm}(\theta', \phi')$$

The multipole expansion allows us to parametrize the charge distribution in terms of its multipole moments. Further, the contribution of a moment q_{lm} to the potential falls off as $1/r^{l+1}$. Therefore, at large distances from a localized charge distribution, only a few non-zero multipole moments with the lowest values of l make significant contributions to Φ and are relevant. The remaining moments could be neglected. This allows us to parametrize even complicated charge distributions in terms of a few lowest l mutipole moments. The condition under which this approximation is valid is that the distance to the observation point (at which Φ is measured) is much larger as compared to the size of the charge distribution.

b) For a spherically symmetric charge distribution, $\rho(\vec{x}) \equiv \rho(r, \theta, \phi) = \rho(r)$, independent of the angular variables. Therefore we can write the multipole moments as a product of two integrals,

$$q_{lm} = \int_0^\infty r'^2 dr'\rho(r')r'^l \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' Y^*_{lm}(\theta', \phi')$$

Since $Y_{00}(\theta', \phi') = 1/\sqrt{4\pi}$, we can insert $Y_{00}(\theta', \phi')\sqrt{4\pi} = 1$ in the angular integration. Now, from the orthogonality property of spherical harmonics it follows that the angular integral is proportional to δ_{l0} and hence vanishes for all $l \geq 1$ ($l = 0$ being the monopole moment).

c) In the case of axial symmetry about the z-axis, ρ is independent of the azimuthal coordinate ϕ. In this case,

$$q_{lm} = (const) \int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \rho(r', \theta')r'^l P_l^m(\cos\theta') \int_0^{2\pi} d\phi' e^{-im\phi'}$$

where we have used the fact that $Y_{lm}(\theta, \phi) = (const)P_l^m(\cos\theta)e^{im\phi}$. Now, the ϕ integral gives a δ_{m0}. Thus the only non-vanishing moments in this case are q_{l0}.

d) In this case the charge density is given by $\rho = q\delta(x')\delta(y')\left(\delta(z' - a) - \delta(z' + a)\right)$. The three components of the dipole moment are q_{1m}, for $m = 1, 0, -1$. Since the problem has axial symmetry about the z-axis, the only non-vanishing component is q_{10} which is now given by (using the expressions for Y_{10}, ρ and noting that $r' = \sqrt{x'^2 + y'^2 + z'^2}$)

$$\begin{aligned} q_{10} &= \int d^3x'\rho(\vec{x}\,')r' Y^*_{10}(\theta', \phi') = \sqrt{\frac{3}{4\pi}} \int dx' \int dy' \int dz' \\ &\qquad \times q\delta(x')\delta(y')\left(\delta(z' - a) - \delta(z' + a)\right)\sqrt{x'^2 + y'^2 + z'^2}\cos\theta \\ &= \sqrt{\frac{3}{4\pi}} \int dz' q\left(\delta(z' - a) - \delta(z' + a)\right)|z'| = aq\sqrt{\frac{3}{\pi}} \end{aligned}$$

Note that $\delta(x')$ and $\delta(y')$ force $\vec{x}\,'$ to be in the z direction and hence, $\theta = 0$.

4. (a) Discuss the consistency of the magnetostatic equation $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$ with the continuity equation $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$.

   (b) Show that the work done by a magnetic field $\vec{B}$ on a charged particle, moving with velocity $\vec{v}$ under the influence of $\vec{B}$, is zero.

   (c) Starting with the magnetostatic equation given in part (a) derive Ampere's law for a stright conducting wire carrying current I.

Solution (points: 5+4+4)
a) The magnetostatic equation implies that $\vec{\nabla} \cdot \vec{J} = \frac{c}{4\pi}\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$. *Hence, it is consistent with the continuity equation only when* $\frac{\partial \rho}{\partial t} = 0$. *On the other hand, from the Gauss law equation it follows that* $\frac{\partial \rho}{\partial t} = \frac{1}{4\pi}\vec{\nabla} \cdot (\frac{\partial \vec{E}}{\partial t})$. *Therefore, if the magnetostatic equation is modified to* $\vec{\nabla} \times \vec{B} - \frac{1}{c}(\frac{\partial \vec{E}}{\partial t}) = \frac{4\pi}{c}\vec{J}$, *it becomes fully consistent with the continuity equation.*

b) The elemental work done on a charged particle of velocity $\vec{v}$ *moving in a magnetic field is* $dW = \vec{F} \cdot \vec{dx} = \frac{q}{c}(\vec{v} \times \vec{B}) \cdot \vec{dx}$. *But in this case,* $\vec{dx} = \vec{v}dt$ *and since* $\vec{v} \times \vec{B}$ *is perpendicular to* $\vec{v}$ *(and of course also to* $\vec{B}$*), it follows that* $(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$.

c) On a plane perpendicular to the current carrying conductor consider a disc of radius r *centred at the conductor. Integrate the magnetostatic equation over the area of this disc. Then* $\int d\vec{S} \cdot J = I$ *and* $\int d\vec{S} \cdot (\vec{\nabla} \times \vec{B}) = \oint \vec{B} \cdot \vec{dl} = 2\pi r B_\phi$, *where,* B_ϕ *is the component of* $\vec{B}$ *tangent to the boundary of the disc. From the symmetry of the problem, this clearly is the only non-vanishing component of* $\vec{B}$. *Hence,* $\vec{B} = (2I/rc)\hat{\phi}$.

5. (a) Starting from Maxwell equations, derive the wave equation satisfied by the vector potential $\vec{A}$ in the Lorenz gauge. (You may need the vector identity$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})$)

   (b) The equation for the vector potential $\vec{A}$ in the Lorenz gauge and in the presence of a current source has a solution

$$\vec{A}(\vec{x}, t) = \frac{1}{c} \int d^3x' \frac{\left[\vec{J}(\vec{x}', t')\right]_{ret}}{|\vec{x} - \vec{x}'|}$$

   in terms of the retarded time $t' = t - |\vec{x} - \vec{x}'|/c$. For a sinusoidal source term, $\vec{J}(\vec{x}, t) = \vec{J}(\vec{x})e^{-i\omega t}$, write down and discuss the nature of the solution in the "near zone" and the "far zone" approximations.

Solution (points: 6+7)
a) Start with the Maxwell equation containing the source term $\vec{J}$ *and substitute for the electric and magnetic fields in terms of the potentials,* $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}(\partial \vec{A}/\partial t)$ *and* $\vec{B} = \vec{\nabla} \times \vec{A}$. *This gives*

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t}\right) = -\frac{4\pi}{c}\vec{J}$$

On imposing the Lorenz gauge condition $\vec{\nabla}\cdot\vec{A}+\frac{1}{c}(\partial\Phi/\partial t)=0$ one gets the desired equation,

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}\vec{\nabla}=-\frac{4\pi}{c}\vec{J}$$

b) We know that $\left[\vec{J}(\vec{x}',t')\right]_{ret}=\vec{J}(\vec{x}',t'=t-|\vec{x}-\vec{x}'|/c)$, so for the given sinusoidal current,

$$\vec{A}(\vec{x},t)=\frac{e^{-i\omega t}}{c}\int d^3x'\,\frac{\vec{J}(\vec{x}')\,e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

where $k=\omega/c=2\pi/\lambda$. There are three length scales in the problem: 1) the linear extension of the current distribution denoted by d (then, with the origin of the coordinate system chosen within the current distribution, one has $x'\lesssim d$), 2) the length λ which is the distance that a signal travels during one oscillation of the source (note that $2\pi/\omega=T$ is the time period of the oscillating source), 3) the distance to the observer denoted by $x=|\vec{x}|$. For a well localized source, we always assume that $d<<x,\lambda$. Now, the "near zone" is characterized by , $d<<x<<\lambda$. We then make the approximation $k|\vec{x}-\vec{x}'|\sim k|\vec{x}|<<1$ or $e^{ik|\vec{x}-\vec{x}'|}\sim 1$, so that

$$\vec{A}(\vec{x},t)=\frac{e^{-i\omega t}}{c}\int d^3x'\,\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Except for the overall time modulation, this has the character of a magnetostatic field. The "far zone" is characterized by , $d<<\lambda<<x$. Then we can make the approximation $|\vec{x}-\vec{x}'|\sim x-\vec{x}\cdot\vec{x}'/x$ and $1/|\vec{x}-\vec{x}'|\rightarrow 1/x$, leading to

$$\vec{A}(\vec{x},t)=\frac{1}{c}\frac{e^{i(kr-\omega t)}}{x}\int d^3x'\,\vec{J}(\vec{x}')\,e^{-ik\vec{x}\cdot\vec{x}'/x}$$

The factor in front of the integral shows that this has the character of an expanding spherical wave.

6. (a) Consider the linear transformation $\tilde{x}^\mu=L^\mu{}_\nu x^\nu$. Find the constraint that the invariance of the space-time interval $(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2$ imposes on the matrix L, showing the steps in your calculation clearly.

   (b) If a 4-vector V^μ transforms as a contravariant vector under Lorentz transformations, work out the transformation of $V_\mu=\eta_{\mu\rho}V^\rho$.

   (c) Show that the two Maxwell equations with sources are contained in the relativistic expression

$$\partial_\mu F^{\mu\nu}=\frac{4\pi}{c}J^\nu$$

Solution (points:(5+4+5))

a) In matrix notation, the space-time interval can be written as $(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2=x^T\eta x$, where,

$$\eta=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\qquad x=\begin{pmatrix}x^0\\x^1\\x^2\\x^3\end{pmatrix},\qquad x^T=\begin{pmatrix}x^0&x^1&x^2&x^3\end{pmatrix}$$

The linear transformation takes the form $\tilde{x} = Lx$, where L is the 4×4 matrix with elements $L^\mu{}_\nu$ (with μ running over rows and ν running over the columns of the matrix). For x^T the transformation reads $\tilde{x}^T = x^T L^T$. The invariance of the interval means that $x^T \eta x = \tilde{x}^T \eta \tilde{x}$ from which the constraint on L follows as

$$L^T \eta L = \eta$$

b) In matrix notation the relation between V_μ and V^μ can be written as $V_\mu = \eta_{\mu\rho} V^\rho = (\eta V)_\mu$ where the column matrix V is constructed from the components of the contravariant vector. After the transformation, we have $\tilde{V}_\mu = (\eta \tilde{V})_\mu$ where, $\tilde{V} = LV$. Hence we have, $\tilde{V}_\mu = (\eta L V)_\mu = (\eta L \eta^{-1} \eta V)_\mu$ or,

$$\tilde{V}_\mu = (\eta L \eta^{-1})_\mu{}^\nu V_\nu$$

This is the transformation of a contravariant vector.

c) We start by writing the relativistic expression seperately for $\nu = 0$ and $\nu = j$ (where j is a space index),

$$\partial_i F^{i0} = \frac{4\pi}{c} J^0 \qquad \partial_0 F^{0j} + \partial_i F^{ij} = \frac{4\pi}{c} J^j$$

where we have used $F^{00} = 0$. Now we note that $J^0 = c\rho$, $F^{i0} = E^i$, $F^{ij} = \epsilon^{ji}{}_k B^k$ and $\partial_i F^{ij} = \epsilon^{ji}{}_k \partial_i B^k = (\vec{\nabla} \times \vec{B})^j$. Using these we recover the two Maxwell equations,

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho\,, \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{J}$$