Final Examination Paper for Electrodynamics-I

Date: *Thursday, Jan 04, 2007,* Time: *09:00 - 15:00* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks*

`Note: Please explain your reasoning clearly`

Questions:	1	2	3	4	5	6	Total
Marks:	15	10	15	10	15	15	80

1. (a) Consider the potential $\Phi(\vec{x})$ at $\vec{x}$ outside a localized charge distribution $\rho(\vec{y})$. Obtain the contribution to $\Phi(\vec{x})$ of the first three multipole moments of $\rho(\vec{y})$.

   (b) Consider a potential problem which is invariant under rotations around the z-axis (that is, with azimuthal symmetry). Assume that we are able to expand the potential along a line that makes an angle θ_0 with the z-axis as,

$$\Phi(r,\theta_0) = \sum_{l=0}^{\infty} \frac{A_l(\theta_0)}{r^{l+1}}$$

   Write down the solution for arbitrary angle θ in terms of the coefficients $A_l(\theta_0)$, using the uniqueness of such expansions.

Solution

a) The potential in this problem is given by $\Phi(\vec{x}) = \int d^3y \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}$. *Using the binomial expansion of* $1/|\vec{x}-\vec{y}|$ *for* $|\vec{x}| >> |\vec{y}|$, *gives*

$$\Phi(\vec{x}) = \frac{q}{x} + \frac{\vec{p}\cdot\vec{x}}{x^3} + \frac{1}{2}\sum_{ij} Q_{ij}\frac{x^i x^j}{x^5} + \cdots$$

where q, $\vec{p}$ *and* Q_{ij} *are the first 3 multipole moments of the charge distribution given by*

$$q = \int d^3y\rho(\vec{y})\,, \qquad \vec{p} = \int d^3y\rho(\vec{y})\vec{y}\,, \qquad Q_{ij} = \int d^3y\rho(\vec{y})(3y_iy_j - y^2\delta_{ij})\,,$$

(This problem can also be solved by using the expansion of $1/|\vec{x}-\vec{y}|$ *in the spherical polar coordinate system)*

b) Because of azimuthal symmetry the general form of the potential is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(C_l r^l + D_l r^{-l-1}\right) P_l(\cos\theta)$$

Comparing with the expansion for $\theta=\theta_0$ *gives* $C_l = 0$ *and* $D_l = A_l(\theta_0)/P_l(\cos\theta_0)$ *so that,*

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\frac{A_l(\theta_0)}{r^{l+1}}\frac{P_l(\cos\theta)}{P_l(\cos\theta_0)}$$

2. (a) Derive the expression for the potential energy of a dipole in an electric field.

(b) In a dielectric medium, if $\vec{P}$ denotes the polarization (dipole moment density per unit volume), show that $-\nabla \cdot \vec{P}$ is the polarization charge density.

Solution

a) Let us assume that the dipole is made of two charges q and $-q$ placed a small distance $\vec{l}$ apart, in the limit $\vec{l} \to 0$ keeping $\vec{p} = q\vec{l}$ fixed. In an electric field $\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x})$, the potential energy of the system, before taking the limit, is $q\Phi(\vec{x} + \frac{1}{2}\vec{l}) - q\Phi(\vec{x} - \frac{1}{2}\vec{l})$, where $\vec{x}$ is the position of the center of the charge system and $\vec{l}$ is taken to point from $-q$ to q. Using the Taylor expansion $\Phi(\vec{x}+\frac{1}{2}\vec{l}) = \Phi(\vec{x})+\frac{1}{2}\vec{l}\cdot\vec{\nabla}\Phi(\vec{x})+\cdots$ and taking the limit $\vec{l} \to 0$, one gets the dipole potential energy as $-\vec{p}\cdot\vec{E}(\vec{x})$.

b) Start with the expression for the electrostatic potential at $\vec{x}$ due to a dipole moment density in a volume ΔV around a point $\vec{y}$, $\Delta\Phi(\vec{x}) = \Delta V \vec{P}(\vec{y}) \cdot (\vec{x} - \vec{y})/|\vec{x} - \vec{y}|^3$. The total potential at $\vec{x}$ due to the polarized medium is then,

$$\Phi(\vec{x}) = \int_V d^3y\, \vec{P}(\vec{y}) \cdot \vec{\nabla}_{(y)}\left(\frac{1}{|\vec{x} - \vec{y}|}\right) = -\int_V d^3y\, \frac{\vec{\nabla}_{(y)} \cdot \vec{P}(\vec{y})}{|\vec{x} - \vec{y}|}$$

where we have dropped a surface term arising from an integration by parts. Now, using $\nabla^2_{(x)}(1/|\vec{x} - \vec{y}|) = -4\pi\delta(\vec{x} - \vec{y})$, one gets

$$\vec{\nabla} \cdot \vec{E}(\vec{x}) = -4\pi\vec{\nabla} \cdot \vec{P}(\vec{x})$$

Comparing with the Maxwell equation $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$, we see that $-\nabla \cdot \vec{P}$ can be regarded as an effective charge density due to the polarization of the medium.

3. (a) Starting from Maxwell equations, derive the wave equation satisfied by the vector potential $\vec{A}$ in the Lorenz gauge.

(b) The equation for the vector potential $\vec{A}$ in the Lorenz gauge and in the presence of a current source has a solution

$$\vec{A}(\vec{x}, t) = \frac{1}{c}\int d^3x' \frac{\left[\vec{J}(\vec{x}', t')\right]_{ret}}{|\vec{x} - \vec{x}'|}$$

in terms of the spherically symmetric retarded Green function. Evaluate this expression for a sinusoidal source term, $\vec{J}(\vec{x}, t) = \vec{J}(\vec{x})e^{-i\omega t}$. Write and discuss the solution in the "near zone" and the "far zone" approximations.

Solution

a) Start with the Maxwell equation containing the source term $\vec{J}$ and substitute for the electric and magnetic fields in terms of the potentials, $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}(\partial\vec{A}/\partial t)$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. This gives

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = -\frac{4\pi}{c}\vec{J}$$

On imposing the Lorenz gauge condition $\vec{\nabla} \cdot \vec{A} + \frac{1}{c}(\partial\Phi/\partial t) = 0$ one gets the desired equation,

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}\vec{\nabla} = -\frac{4\pi}{c}\vec{J}$$

b) We know that $\left[\vec{J}(\vec{x}', t')\right]_{ret} = \vec{J}(\vec{x}', t' = t - |\vec{x} - \vec{x}'|/c)$, so for the given sinusoidal current,

$$\vec{A}(\vec{x}, t) = \frac{e^{-i\omega t}}{c}\int d^3x' \frac{\vec{J}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x} - \vec{x}'|}$$

where $k = \omega/c = 2\pi/\lambda$. Let d denote the linear extension of the current distribution ($x' \lesssim d$). In the near zone, $d << x << \lambda$ where $x = |\vec{x}|$. We then make the approximation $k|\vec{x} - \vec{x}'| << 1$ or $e^{ik|\vec{x}-\vec{x}'|} \sim 1$, so that

$$\vec{A}(\vec{x}, t) = \frac{e^{-i\omega t}}{c}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

Except for the overall time modulation this has the character of a magnetostatic field. In the far zone, $d << \lambda << x$. Then, $|\vec{x} - \vec{x}'| \sim x - \vec{x} \cdot \vec{x}'/x$ and $1/|\vec{x} - \vec{x}'| \to 1/x$, leading to

$$\vec{A}(\vec{x}, t) = \frac{1}{c}\frac{e^{i(kr-\omega t)}}{x}\int d^3x'\, \vec{J}(\vec{x}')\, e^{-ik\vec{x}\cdot\vec{x}'/x}$$

The factor in front of the integral shows that this has the character of an expanding spherical wave.

4. (a) Consider a current distribution with local current density $\vec{J}(\vec{x})$ in a volume V placed in an external magnetic field $\vec{B}(\vec{x})$. Find the expression for the net force $\vec{F}$ acting on the current distribution.

   (b) Starting with

$$\vec{B}(\vec{x}) = \frac{1}{c}\int d^3x'\, \vec{J}(\vec{x}') \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$$

   find the expression for the force acting between two thin wires carrying currents I_1 and I_2 (the wires are not necessarily straight and parallel).

Solution

a) The Lorentz force acting on a volume element d^3x within the current distribution is $d\vec{F} = \dfrac{1}{c}(\rho d^3x)\vec{v} \times \vec{B}$ so that the force on the volume V is $\vec{F} = \frac{1}{c}\int_V \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x$

b) For a wire we can write $d^3x = \vec{ds} \cdot \vec{dl}$ with $\vec{dl}$ along the length of the wire and $\vec{ds}$ a surface element over the cross section of the wire. In the thin wire approximation, $\vec{J}$ is parallel to $\vec{dl}$ and the variation of $\vec{B}$ over the cross section can be neglected. Then, $\int_V d^3x \vec{J}(\vec{x}) \times \vec{B}(\vec{x}) = \int_l \int_S (\vec{ds} \cdot \vec{dl})(\vec{J} \times \vec{B}) = \int_l \int_S (\vec{ds} \cdot \vec{J})(\vec{dl} \times \vec{B}) = I_1 \int_{l_1} \vec{dl} \times \vec{B}$. This gives the force on wire 1 due to the field $\vec{B}$. Also in this approximation and using the same arguments, the expression for the magnetic field produced by a current *distribution in wire 2 (given in the question above) becomes,* $\vec{B}(\vec{x}) = \frac{I_2}{c}\int_{l_2} d\vec{l}'\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$. *Using this in the force expression gives,*

$$F_{12} = \frac{I_1 I_2}{c^2}\int_{l_1}\int_{l_2} d\vec{l}\times d\vec{l}'\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

5. The boundary between two media with different electric and magnetic properties, (ϵ,μ) and (ϵ',μ'), lies along the $x-y$ plane at $z=0$. A plane electromagnetic wave $\vec{E}=\vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ (with a similar expression for $\vec{B}$) traveling in medium (ϵ,μ) is incident on the boundary. For this wave, $k=\sqrt{\epsilon\mu}\,\omega/c$, where $k=|\vec{k}|$. Assume that there is a reflected plane wave characterized by $(\vec{k}'',\omega'')$ in the (ϵ,μ) medium and a refracted one characterized by $(\vec{k}',\omega')$ in the (ϵ',μ') medium.

   (a) Show that $k=k''$ and $k'=k\sqrt{\epsilon'\mu'/\epsilon\mu}$

   (b) Assuming that $\vec{k}$, $\vec{k}'$ and $\vec{k}''$ lie in the same plane together with the normal $\hat{z}$ to the boundary (you do not have to prove this), show that one obtains the laws of reflection and refraction from electromagnetism.

   Solution

   a) The refracted and reflected plane waves are

$$\vec{E}'=\vec{E}_0' e^{i(\vec{k}'\cdot\vec{x}-\omega' t)}\,, \qquad \vec{E}''=\vec{E}_0'' e^{i(\vec{k}''\cdot\vec{x}-\omega'' t)}$$

   where,

$$k'=\sqrt{\epsilon'\mu'}\,\omega'/c\,, \qquad k''=\sqrt{\epsilon\mu}\,\omega''/c$$

   due to the plane wave nature of the solution (we suppress similar equations for the $\vec{B}$ field). At the boundary between the two media, the normal and tangential components of $\vec{E}$, $\vec{E}'$ and $\vec{E}''$ must satisfy appropriate boundary conditions. We do not need the actual form of the boundary conditions here, but it is clear that for these conditions to be satisfied at all times, $\vec{E}$, $\vec{E}'$ and $\vec{E}''$ must have the same time dependence. In other words, $\omega=\omega'=\omega''$. Using this in the expressions for k' and k'' it then follows that $k=k''$ and $k'=k\sqrt{\epsilon'\mu'/\epsilon\mu}$.
   b) The boundary conditions on $\vec{E}$, $\vec{E}'$ and $\vec{E}''$ must also remain valid at all points on the boundary. This means that for any point $\vec{x}$ on the boundary of the two media, we must have $\vec{k}\cdot\vec{x}=\vec{k}'\cdot\vec{x}=\vec{k}''\cdot\vec{x}$ or,

$$kx\cos\theta = k'x\cos\theta' = k''x\cos\theta''$$

   Now from the description of the problem it is clear that the angles of incidence ($\angle i$), refraction ($\angle r'$) and reflection ($\angle r''$) are given by $\hat{k}\cdot\hat{z}=\cos(\angle i)$, $\hat{k}'\cdot\hat{z}=\cos(\angle r')$ and $\hat{k}''\cdot\hat{z}=\cos(\angle r'')$, where $\hat{k}$ is a unit vector along $\vec{k}$, etc. We want to find a relation between the angles θ,θ',θ'' and $\angle i,\angle r',\angle r''$. For the sake of convenience, let us choose the origin of our coordinate system to lie on the boundary between the *media. Then the position vector $\vec{x}$ of any point on the boundary is tangent to the boundary and is normal to $\hat{z}$. For this choice of origin,*

$$\theta = \frac{\pi}{2} - \angle i\,, \qquad \theta' = \frac{\pi}{2} - \angle r'\,, \qquad \theta'' = \frac{\pi}{2} - \angle r''$$

Then, $k\cos\theta = k''\cos\theta''$ implies $\sin\angle i = \sin\angle r''$ or $\angle i = \angle r''$ which is the law of reflection. Similarly, $k\cos\theta = k'\cos\theta'$ implies

$$\frac{\sin\angle i}{\sin\angle r'} = \frac{k'}{k} = \sqrt{\frac{\epsilon'\mu'}{\epsilon\mu}}$$

which is the law of refraction.

6. (a) For a coordinate transformation $x^\mu \to \widetilde{x}^\mu$ define *covariant* and *contravariant* vectors. For the special case of Lorentz transformations, show that if x^μ transforms as a contravariant vector, $\widetilde{x}^\mu = L^\mu{}_\nu x^\nu$, then $x_\mu = \eta_{\mu\nu}x^\nu$ transforms as a covariant vector.

   (b) Consider a Lorentz transformation L relating two reference frames S and $\widetilde{S}$. Assume that an observer in S measures the velocity of a particle as $\vec{u}$. (i) Find the particle velocity $\tilde{u}^i$ in the frame $\widetilde{S}$ in terms of $L^\mu{}_\nu$ and u^i. (ii) When frame $\widetilde{S}$ moves away from S with velocity v in the x^1 direction, check that the relation between $\tilde{u}^i$ and u^i reduces to

$$\tilde{u}^1 = \frac{u^1 - v}{1 - \dfrac{vu^1}{c^2}}\,, \qquad \tilde{u}^2 = \frac{u^2}{\gamma\left[1 - \dfrac{vu^1}{c^2}\right]}\,, \qquad \tilde{u}^3 = \frac{u^3}{\gamma\left[1 - \dfrac{vu^1}{c^2}\right]}$$

Solution

a) In general, covariant vectors W_μ and contravariant vectors V^μ transform as

$$\widetilde{W}_\mu(\widetilde{x}) = \frac{\partial x^\nu}{\partial \widetilde{x}^\mu} W_\nu(x)\,, \quad \widetilde{V}^\mu(\widetilde{x}) = \frac{\partial \widetilde{x}^\mu}{\partial x^\nu} V^\nu(x)$$

For the Lorentz transformation $\widetilde{x}^\mu = L^\mu{}_\nu x^\nu$, we get $\frac{\partial \widetilde{x}^\mu}{\partial x^\nu} = L^\mu{}_\nu$ and $\frac{\partial x^\nu}{\partial \widetilde{x}^\mu} = (L^{-1})^\nu{}_\mu$ so that $\widetilde{V}^\mu(\widetilde{x}) = L^\mu{}_\nu V^\nu(x)$ and $\widetilde{W}_\mu(\widetilde{x}) = (L^{-1})^\nu{}_\mu W_\nu(x)$ or, in matrix notation,

$$\widetilde{V} = LV\,, \qquad \widetilde{W} = (L^{-1})^T W$$

where V and W stand for 4-component column vectors. The defining equation for L, i.e., $L^T\eta L = \eta$ implies that $L^T = \eta L^{-1}\eta^{-1}$ or $(L^T)^{-1} = \eta L \eta^{-1}$. Using this, the Lorentz transformation of a covariant vector becomes, in matrix notation,

$$\widetilde{W} = \eta L \eta^{-1} W$$

Let us now look at the Lorentz transformation of $\eta_{\mu\nu}x^\nu$. In matrix notation, the Lorentz transformed quantity is

$$\eta\widetilde{x} = \eta L x = \eta L \eta^{-1}(\eta x)$$

which is the Lorentz transformation of a covariant vector.
b) (i) If the particle moves distance $\Delta\vec{x}$ *in time* Δt *at velocity* $\vec{u}$ *in the* S *frame, then,* $\Delta x^i = u^i \Delta t = (u^i/c)\Delta x^0$. *In the* $\widetilde{S}$ *frame, the corresponding velocity is given by* $\widetilde{u}^i = c\Delta\widetilde{x}^i/\Delta\widetilde{x}^0$. *Since the Lorentz transformation between the two frames is linear, we have* $\Delta\widetilde{x}^\mu = L^\mu{}_\nu \Delta x^\nu$ *which gives,*

$$\Delta\widetilde{x}^i = L^i{}_j \Delta x^j + L^i{}_0 \Delta x^0 = (L^i{}_j u^j/c + L^i{}_0)\Delta x^0$$

and

$$\Delta\widetilde{x}^0 = L^0{}_j \Delta x^j + L^0{}_0 \Delta x^0 = (L^0{}_j u^j/c + L^0{}_0)\Delta x^0$$

From this we get the transformation of velocity as

$$\frac{\widetilde{u}^i}{c} = \frac{L^i{}_j u^j/c + L^i{}_0}{L^0{}_j u^j/c + L^0{}_0}$$

(ii) For motion confined to the x^1 *direction, the only non-zero components of* L *are* $L^0{}_0 = L^1{}_1 = \gamma$, $L^0{}_1 = L^1{}_0 = -\gamma\beta$ *and* $L^2{}_2 = L^3{}_3 = 1$. *Using these values one recovers the desired form of velocity transformation.*