Final Examination Paper for Electrodynamics-I

Date: *Friday, Nov 03, 2006*, Time: *12:30 - 18:30* **[Solutions]**

Allowed help material: *Physics and Mathematics handbooks*

`Note: Please explain your reasoning clearly`

Questions:	1	2	3	4	5	6	Total
Marks:	10	15	15	15	10	15	80

1. (a) Show that if $\nabla \times \vec{E} = 0$, then one can always find a function Φ such that $\vec{E} = -\nabla\Phi$.

   (b) If $\vec{E}$ is the electric field due to a charge distribution of density ρ, what is the equation obeyed by Φ? For a localized charge distribution , write the solution for Φ in the absence of boundaries.

   Solution

   a) This is worked out in the lecture notes "Rapid Review of Vector Calculus" section 1.4. The strategy is as follows: Stokes's theorem leads to $\oint_l \vec{E} \cdot \vec{dl} = 0$ for a closed path l, which shows the path independence of the integral. To see this, split the path l into two parts, $l = l_{AB} + l_{BA}$. Keeping l_{BA} fixed and varying l_{AB} one sees that $\int_{r_A}^{r_B} \vec{E} \cdot dl (= -\int_{r_B}^{rA} \vec{E} \cdot dl)$ remains unchanged which implies that it can only depend on the end points of the path $(\vec{r}_A, \vec{r}_B)$ and not on the actual shape of the path connecting these two points, that is, $\int_{r_A}^{r_B} \vec{E} \cdot \vec{dl} = \Phi(\vec{r}_A, \vec{r}_B)$. A small variation of the point B, $\vec{r}_B \to \vec{r}_B + \delta\vec{r}_B$ results in $\int_{r_A}^{r_B+\delta r_B} \vec{E} \cdot \vec{dl} = \Phi(\vec{r}_A, \vec{r}_B + \delta r_B)$. Comparing the two equations one sees that for a small δr_B, the variation of the left hand side is $\vec{E} \cdot \delta\vec{r}_B$ while that of the right hand side is $\vec{\nabla}\Phi \cdot \delta\vec{r}_B$. Since this is true for any small $\delta\vec{r}_B$, one concludes that $\vec{E} = -\nabla\phi$ (the extra negative sign introduced here is a matter of convention and does not change the result).

   b) In this case, $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ so that $\nabla^2\Phi = -4\pi\rho$. Since ρ corresponds to a localized charge distribution, and there are no boundary effects, the solution for Φ is $\Phi(\vec{x}) = \int d^3x'\rho(\vec{x}')/|\vec{x}-\vec{x}'|$. This can be verified using $\nabla^2(1/|\vec{x}-\vec{x}'|) = -4\pi\delta^3(\vec{x}-\vec{x}')$.

2. Consider an external electric field given by $E_i = C_i + D_{ij}x^j$ in a region of space free of charges and currents.

   (a) Show that the matrix D_{ij} is traceless ($\sum_i D_{ii} = 0$) and symmetric ($D_{ij} = D_{ji}$). What is the external potential $\Phi_{ext}(\vec{x})$ corresponding to this electric field (ignore the undetermined constant piece)?

   (b) In this external field place a *conducting* sphere of radius R centered at $\vec{x} = 0$ and carrying zero net charge. Suppose the polarization of the sphere in the external field is described by a dipole moment p_i and a quadrupole moment Q_{ij}. Write the expression for the induced potential $\Phi_{in}(\vec{x})$ for $|\vec{x}| \geq R$ generated by these multipole moments. What is the total potential $\Phi_{ext} + \Phi_{in}$ inside the sphere ($|\vec{x}| \leq R$)?

(c) Determine p_i and Q_{ij} in terms of C_i, D_{ij} and R and find the total potential $\Phi_{ext} + \Phi_{in}$ outside the sphere ($|\vec{x}| \geq R$).

(d) Compute the induced surface charge density on the sphere (Hint: In spherical coordinates one can write, $x^i = x\hat{x}^i$ where $\hat{x}^i$ are the Cartesian components of the radial unit vector $\hat{x}$, *e.g.*, $\hat{x}^3 = \cos\theta$, $\hat{x}^1 = \sin\theta\cos\phi$, $\hat{x}^2 = \sin\theta\sin\phi$. Hence, they do not vary with radial distance x).

Solution

a) The electric field satisfies $\vec{\nabla} \cdot \vec{E} = \sum_i \partial_i E^i = 0$ *implying* $\sum_i D_{ii} = 0$ *and* $(\vec{\nabla} \times \vec{E})_i = \sum_{jk} \epsilon_i{}^{jk} \partial_j E_k = 0$ *implying* $\sum_{jk} \epsilon_i{}^{jk} D_{jk} = 0$. *Therefore, the matrix* D *is traceless and symmetric. The corresponding potential is* $\Phi_{ext} = -\sum_i C_i x^i - \frac{1}{2}\sum_{ij} D_{ij} x^i x^j$, *consistent with* $\vec{E} = -\nabla\Phi_{ext}$.

b) The induced potential for $|\vec{x}| \geq R$ *due to the polarized sphere is that due a dipole of moment* $\vec{p}$ *and a quadrupole of moment matrix* Q_{ij},

$$\Phi_{in} = \frac{\vec{p} \cdot \vec{x}}{x^3} + \frac{1}{2}\frac{Q_{ij} x^i x^j}{x^5}$$

The total potential $\Phi_{ext} + \Phi_{in}$ *inside the sphere is zero (up to a constant).*

c) The sphere being conducting, the total potential $\Phi_{in} + \Phi_{ext}$ *on its surface must vanish,*

$$\left(\frac{p_i x^i}{R^3} + \frac{1}{2}\frac{Q_{ij} x^i x^j}{R^5} - C_i x^i - \frac{1}{2} D_{ij} x^i x^j\right)\Big|_{|\vec{x}|=R} = 0$$

Comparing terms with the same tensor structure, one gets $p_i = R^3 C_i$ *and* $Q_{ij} = R^5 D_{ij}$. *The total potential for* $|\vec{x}| \geq R$ *is*

$$\Phi = \Phi_{in} + \Phi_{ext} = C_i x^i \left(\frac{R^3}{x^3} - 1\right) + \frac{1}{2} D_{ij} x^i x^j \left(\frac{R^5}{x^5} - 1\right)$$

d) Using the notation described in the question, one can write the total potential ϕ *in spherical polar coordinates as*

$$\Phi = C_i \hat{x}^i \left(\frac{R^3}{x^2} - x\right) + \frac{1}{2} D_{ij} \hat{x}^i \hat{x}^j \left(\frac{R^5}{x^3} - x^2\right)$$

where $\hat{x}^i$ *are independent of* $x = |\vec{x}|$, *depending only on the angular variables. The surface charge density is given by* $(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\sigma$ *where* $\hat{n}$ *is the unit normal to the surface of the sphere. In this case,* $\vec{E}_1 = 0$ *and* $\vec{E}_2 \cdot \hat{n} = -\partial\Phi/\partial x|_{x=R}$. *Therefore,*

$$\sigma = -\frac{1}{4\pi}\frac{\partial\Phi}{\partial x}\Big|_{x=R} = \frac{1}{4\pi}\left(3C_i \hat{x}^i - \frac{5}{2} R D_{ij} \hat{x}^i \hat{x}^j\right)$$

3. (a) Consider a surface on which an electric current of surface current density $\vec{K}$ flows. Assuming steady state, work out the behaviour of the normal and tangential components of the magnetic field $\vec{B}$ across the surface. Argue that this result remains valid even when the steady state condition is relaxed.

(b) In a magnetic medium, what is the significance of the field $\vec{H}$ in comparison to the field $\vec{B}$? Briefly describe the phenomenon of *magnetic hysteresis.*

Solution

a) In steady state the magnetic field satisfies $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$. First, draw a small, so called, "Gaussian pill-box" of height h across the surface. The top and bottom faces of the pill-box have areas ΔS each and are parallel to the surface. Denote the value of the magnetic field on the bottom face of the box by $\vec{B}_1$ and on the top face of the box by $\vec{B}_2$. The unit normals to these faces are $\hat{n}_1$ and $\hat{n}_2$ ($\hat{n}_2 = -\hat{n}_1 = \hat{n}$). Then,

$$0 = \lim_{h\to 0} \int_{pill-box} d^3x \vec{\nabla} \cdot \vec{B} = \lim_{h\to 0} \int_{\partial(pill-box)} \vec{ds} \cdot \vec{B} = (\vec{B}_2 \cdot \hat{n}_2 + \vec{B}_1 \cdot \hat{n}_1)\Delta S$$

where the contribution from the sides have dropped in the limit $h \to 0$. Hence we have,

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0$$

Let us now replace the pill-box by a rectangular loop that has its longer sides of length Δl parallel to the surface and its shorter sides of height h perpendicular to the surface and going through it. A unit vector along the lower side of the rectangle is $\hat{t}_1$ and one along the upper side is $\hat{t}_2$, both being parallel to the surface. Picking an orientation along the loop, one has $\hat{t}_2 = -\hat{t}_1 = \hat{t}$. Integrate $\vec{\nabla} \times \vec{B}$ over the loop area,

$$\int_{loop\ area} \vec{\nabla} \times \vec{B} \cdot \vec{ds} = \frac{4\pi}{c} \int_{loop\ area} \vec{J} \cdot \vec{ds}$$

In the limit of $h \to 0$, the left hand side gives,

$$\lim_{h\to 0} \int_{loop} \vec{B} \cdot \vec{dl} = (\vec{B}_2 \cdot \hat{t}_2 + \vec{B}_1 \cdot \hat{t}_1)\Delta l$$

The right and side gives the total current flowing through the loop area. In the limit $h \to 0$, this is simply the surface current over the length Δl of the squashed loop. A unit vector normal to the loop area (it i.e., along $\vec{ds}$ is) $\hat{t}' = \hat{n} \times \hat{t}$. Hence,

$$\lim_{h\to 0} \int_{loop\ area} \vec{J} \cdot \vec{ds} = \vec{K} \cdot \hat{t}'\Delta l$$

and therefore,

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{t} = \frac{4\pi}{c}\vec{K} \cdot \hat{t}'$$

This is true for all orientations of the loop, or equivalently, for all unit tangent vectors $\hat{t}$ to the surface. The above relation can be rewritten in a form that is *independent of the choice of a tangent vector* $\hat{t}$ *by noting that* $\hat{t} = \hat{t}' \times \hat{n}$. *Then using the cyclic property of the triple product,* $(\vec{B}_2 - \vec{B}_1) \cdot (\hat{t}' \times \hat{n}) = \hat{t}' \cdot (\hat{n} \times (\vec{B}_2 - \vec{B}_1))$ *and the fact that the equation is valid for all orientations of the loop (that is, all* $\hat{t}'$ *tangent to the surface), one has the alternate form*

$$\hat{n} \times (\vec{B}_2 - \vec{B}_1) = \frac{4\pi}{c} \vec{K}$$

When the steady state condition is relaxed, $\vec{\nabla} \cdot \vec{B} = 0$ *still holds and therefore the behaviour of the normal component of* $\vec{B}$ *does not change. However the second equation is modified to* $\vec{\nabla} \times \vec{B} - \frac{1}{c} \partial \vec{E} / \partial t = \frac{4\pi}{c} \vec{J}$. *This could give an extra contribution to the equation for the tangential component of* $\vec{B}$ *coming from*

$$lim_{h \to 0} \int_{loop\ area} \frac{\partial \vec{E}}{\partial t} \cdot \vec{ds}$$

But in the limit, the loop area (integration domain) shrinks to zero size while the integrand is finite and as a result the integral vanishes. Hence the steady state results are also applicable to non-steady state situations.
b) In a magnetic medium atoms and molecules could carry magnetic dipole moments. In an external magnetic field, these microscopic magnetic dipoles tend to realign themselves giving rise to a net magnetic field that is characterized by the magnetic moment density or magnetization $\vec{M}$ *of the medium. The total magnetic field we measure is a sum of the field produced by* $\vec{M}$ *and the externally applied field. This we denote by* $\vec{B}$. *The field* $\vec{H}$ *is related to these by* $\vec{H} = \vec{B} - 4\pi \vec{M}$ *and it can be shown to satisfy* $\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{J}$ *where* $\vec{J}$ *is the external "free" current density. It is then clear that* $\vec{H}$ *is the part of the magnetic field that is entirely due to the external currents and has no contribution from the response of the medium.*

Magnetic Hysteresis: Since H *is the part of the total magnetic field that can be tuned externally, it is natural to regard* B *as a function of* H. *For diamagnetic and paramagnetic material the relation between the two is linear to a good degree of approximation,* $B = \mu H$. *However, ferromagnetic material respond strongly to the applied field and the relation* $B = B(H)$ *has the following features: Initially, as* H *is increased from zero,* B *also increases due to the fact that more and more microscopic magnetic dipoles in the material realign themselves along the applied field, enhancing its strength. But when most of the microscopic dipoles have realigned, the increase in* B *slows down and the* B *vs* H *curve flattens out. Now, as* H *is decreased,* B *also decreases, but it starts lagging behind* H *since the microscopic dipoles resist flipping their orientation. Even when* H *is reduced to zero, there is still a residual* B *field. As* H *is made negative,* B *reduces further and finally vanishes for some negative* H. *Beyond this,* B *increases in the negative direction until most of the microscopic dipoles are aligned in the new direction and beyond some negative* H *the curve again flattens out. In this way, as* H *is varied sinusoidally,* B *traces a closed loop in the* $B - H$ *plane called the Hysteresis curve (it can be found in any standard textbook on electromagnetism) and the phenomenon is referred to as hysteresis.*

4. A straight wire of length L and radius a has a resistance R and carries current I.

   (a) Find the electric and magnetic fields on the surface of the wire and indicate their directions.

   (b) Evaluate the energy carried into the wire by the above electric and magnetic fields.

   (c) What happens to this energy in the steady state? Verify your answer using the Poynting theorem in the thin wire approximation.

Solution

a)The electric field on the surface is given by $E = \Phi/L$ where the constant potential difference Φ is given by Ohm's law, $\Phi = IR$. Hence, $E = IR/L$. The direction of $\vec{E}$ is parallel to the current and hence to the wire. The magnetic field on the surface is given by Ampere's law as $B = 2I/ca$ (This is obtained by integrating $\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{J}$ over a cross section of the wire and using the cylindrical symmetry of the problem). The direction of $\vec{B}$ is given by the "right-hand-rule" which makes it perpendicular to both $\vec{E}$ and the radius vector of the cylindrical wire. Hence the direction of $\vec{B}$ is along the angular direction of the cylinder.

b) The energy carried into the wire by electric and magnetic fields is the surface integral of the Poynting vector over the surface of the wire. The Poynting vector is $\vec{S} = (c/4\pi)\vec{E} \times \vec{B}$. Since $\vec{E}$ is perpendicular to $\vec{B}$, we have, for the magnitude of the Poynting vector, $S = I^2R/(2\pi aL)$. $\vec{S}$ is directed radially inward, $\vec{S} = -\hat{r}\, S$. The flux $\int \vec{ds} \cdot \vec{S}$ evaluated over the surface of a segment of length L of the cylindrical wire receives contributions only from the curved side-area of the cylinder (of area $2\pi aL$) and not from the top and bottom caps (since then $\vec{S}$ is perpendicular to $\vec{ds}$). Hence the total flux is

$$\int \vec{ds} \cdot \vec{S} = -I^2R$$

The sign is due to fact that $\vec{S}$ is directed radially inward while, for the cylinder, $\vec{ds}$ is directed radially outward. Physically, the negative sign signifies that energy enters into the volume under consideration, rather than leave it.

c) Since in this problem the electric and magnetic fields are constant, the Poynting theorem reduces to

$$-\int_V d^3x \vec{E} \cdot \vec{J} = \int_{\partial V} \vec{ds} \cdot \vec{S}$$

The left hand side is recognized as the expression for the energy injected into the current distribution by the electric and magnetic fields. Thus the energy carried into the wire by the Poynting vector is fully converted into the kinetic energy of the charge carriers. Since the current is constant, the system is in steady state and the extra kinetic energy acquired by the charges is dissipated into heat as a result of collisions within the resistive medium. The left hand side can be computed in the thin wire approximation. For our wire, $d^3x = \vec{dl} \cdot \vec{ds}$ where $\vec{dl}$ is along the length of the wire and the $\vec{ds}$ integration is over the cross sectional area of the wire. In the *thin wire approximation, $\vec{J}$ is parallel to $\vec{dl}$, so that $(\vec{dl}\cdot\vec{ds})(\vec{J}\cdot\vec{E}) = (\vec{J}\cdot\vec{ds})(\vec{dl}\cdot\vec{E})$. Moreover, $\vec{E}$ can be taken to be constant over the cross section of the thin wire. Then*

$$-\int_V d^3x \vec{E}\cdot\vec{J} \approx -(\int \vec{J}\cdot\vec{ds})(\int \vec{dl}\cdot\vec{E}) = -I\Phi = -I^2R$$

which verifies the result of part b) of the question.

5. (a) Show that Maxwell's equations are consistent with the conservation of electric charge.

   (b) The equation for the vector potential $\vec{A}$ in the Lorenz gauge and in the presence of a current source has a solution

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \frac{\left[\vec{J}(\vec{x}',t')\right]_{ret}}{|\vec{x}-\vec{x}'|}$$

   in terms of the spherically symmetric retarded Green function. Evaluate this expression for a sinusoidal source term, $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$. Write and discuss the solution in the "near zone" and the "far zone" approximations.

Solution

a) Start with the two Maxwell equations containing the sources ρ and $\vec{J}$. Taking a time derivative of the first equation and a divergence of the second, one gets $\partial\rho/\partial t + \nabla\cdot\vec{J} = 0$ which is a statement of charge conservation.

b) We know that $\left[\vec{J}(\vec{x}',t')\right]_{ret} = \vec{J}(\vec{x}', t' = t - |\vec{x}-\vec{x}'|/c)$, so for the given sinusoidal current,

$$\vec{A}(\vec{x},t) = \frac{e^{-i\omega t}}{c}\int d^3x' \frac{\vec{J}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

where $k = \omega/c = 2\pi/\lambda$. Let d denote the linear extension of the current distribution ($x' \lesssim d$). In the near zone, $d << x << \lambda$ where $x = |\vec{x}|$. We then make the approximation $k|\vec{x}-\vec{x}'| << 1$ or $e^{ik|\vec{x}-\vec{x}'|} \sim 1$, so that

$$\vec{A}(\vec{x},t) = \frac{e^{-i\omega t}}{c}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

Except for the overall time modulation this has the character of a magnetostatic field. In the far zone, $d << \lambda << x$. Then, $|\vec{x}-\vec{x}'| \sim x - \vec{x}\cdot\vec{x}'/x$ and $1/|\vec{x}-\vec{x}'| \to 1/x$, leading to

$$\vec{A}(\vec{x},t) = \frac{1}{c}\frac{e^{i(kr-\omega t)}}{x}\int d^3x'\, \vec{J}(\vec{x}')\, e^{-ik\vec{x}\cdot\vec{x}'/x}$$

The factor in front of the integral shows that this has the character of an expanding spherical wave.

6. (a) In relativistic electrodynamics, the Lorentz force law is contained in

$$m_0\frac{dU^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}\, U_\nu$$

   where $U^\mu = (\gamma c, \gamma\vec{u})$ is the relativistic 4-velocity, τ is time in the rest-frame of the moving charge ($dt = \gamma d\tau$) and $\gamma^{-1} = \sqrt{1-u^2/c^2}$.

i. Show how this modifies the non-relativistic Lorentz force law, $m(d\vec{u}/dt) = q\vec{E} + (q/c)\vec{u}\times\vec{B}$.

ii. Besides this equation,what other equation is contained in the relativistic force law? Explain its physical significance.

(b) Consider an inertial reference frame $\widetilde{S}$ moving with respect to frame S with velocity v in the positive x direction. If the observer in S measures static charge distribution $\rho(\vec{x})$, find the charge and current distributions as measured by the observer in $\widetilde{S}$. Discuss the non-relativistic limit of your result.

Solution

a) (i) For $\mu = i$, the relativistic equation reduces to

$$m\frac{du^i}{dt} + \frac{m}{\gamma}\frac{d\gamma}{dt}u^i = qF^{i0} - \frac{q}{c}F^{ij}\,u_j$$

But, $F^{i0} = E^i$, $F^{ij} = -\epsilon^{ij}{}_k B^k$ and $(\vec{u}\times\vec{B})^i = -\epsilon^i{}_{jk}u^j B^k$ (with $\epsilon_{123} = 1$), and therefore,

$$m\frac{d\vec{u}}{dt} + \frac{m}{\gamma}\frac{d\gamma}{dt}\vec{u} = q\vec{E} + \frac{q}{c}\vec{u}\times\vec{B}$$

where $m = \gamma m_0$. The relativistic correction is the term involving $d\gamma/dt$. Note that γ being a function of the velocity $\vec{u}$ of the moving particle, is not constant in time. The left hand side has a compact expression in terms of the momentum $\vec{p} = m_0\gamma\vec{u}$ which again makes it resemble a Newtonian force.
(ii) For $\mu = 0$, it reduces to $d(m_0c^2\gamma)/dt = qE^iu^i$. We recognize $\mathcal{E} = m_0c^2\gamma$ as the relativistic energy of the particle. Hence $d(\mathcal{E})/dt = q\vec{u}\cdot\vec{E}$ which gives the power transferred to the charged particle from the electric field.
b) The charge density $\rho(\vec{x})$ and current density $\vec{J}(\vec{x})$ combine into a 4-vector $J^\mu = \{J^0 = c\rho, \vec{J}\}$ which under Lorentz transformations L transforms as

$$\tilde{J}^\mu(\tilde{x}) = L^\mu{}_\nu J^\nu(L^{-1}\tilde{x})$$

In our case, $\vec{J} = 0$ and the non-trivial components of L are, $L^0{}_0 = L^1{}_1 = \gamma$, and $L^1{}_0 = L^0{}_1 = -\gamma\beta$. Therefore the Lorentz transformation gives (suppressing the $\tilde{x}$ dependence)

$$\tilde{\rho} = \gamma\rho\,, \qquad \tilde{J}^1 = -\gamma v\rho\,, \qquad \tilde{J}^2 = \tilde{J}^3 = 0$$

In the non-relativistic limit, $\beta = v/c \to 0$ and $\gamma \to 1$ so that,

$$\tilde{\rho} = \rho\,, \qquad \tilde{J}^1 = -v\rho$$

which is the expected result.