Final Examination Paper for Electrodynamics-I

Allowed help material: *Physics and Mathematics handbooks*
Date: *Saturday, Nov 05, 2005*, Time: *09:00 - 15:00* **[Solutions]**

Questions:	1	2	3	4	5	6	Total
Marks:	10	10	20	10	10	10	70

1. Consider a grounded conducting sphere of radius a the centre of which coincides with the origin of the coordinate system. Place a point charge q at $\vec{y}$ outside it.
   (a) Find the value q' and the position $\vec{y}\,'$ of the image charge inside the sphere.
   (b) Evaluate the potential ϕ at any point $\vec{x}$ outside the sphere.
   (c) Evaluate the surface charge density σ induced on the surface of the sphere.

   Solution
   a) The potential due to q *and* q' *at* $\vec{x}$ *outside the sphere is*

$$\phi(\vec{x}) = \frac{q}{|\vec{x} - \vec{y}|} + \frac{q'}{|\vec{x} - \vec{y}\,'|}$$

   Then, $\phi(|\vec{x}| = a) = 0$ *gives* $q' = -aq/y$ *and* $y' = a^2/y$
   b) The potential at any $\vec{x}$ *is (with* $\hat{x}$ *and* $\hat{y}$ *unit vectors along* $\vec{x}$ *and* $\vec{y}$*, respectively)*

$$\phi(\vec{x}) = \frac{q}{|x\hat{x} - y\hat{y}|} - \frac{aq}{y}\frac{1}{|x\hat{x} - \frac{a^2}{y}\hat{y}|}$$

   c) σ *is related to the discontinuity in the normal component of the electric field across the surface of the sphere:*

$$(\vec{E}_{out} - \vec{E}_{in}) \cdot \hat{x} = 4\pi\sigma$$

   Since $\vec{E}_{in} = 0$ *and* $\vec{E}_{out} = -\vec{\nabla}\phi$*, one gets,*

$$\sigma = -\frac{1}{4\pi}\vec{\nabla}\phi \cdot \hat{x}|_{x=a} = -\frac{1}{4\pi}\frac{\partial\phi}{\partial x}|_{x=a} = -\frac{1}{4\pi}\frac{q}{ay}\frac{1 - a^2/y^2}{(1 + a^2/y^2 - 2\hat{x}\cdot\hat{y}\,a/y)^{3/2}}$$

2. (a) Show that the equation $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{J} = 0$ implies the conservation of charge.
   (b) Starting from the Biot-Savart law,

$$\vec{B}(\vec{x}) = \frac{1}{c}\int d^3x'\,\vec{J}(\vec{x}\,') \times \frac{(\vec{x} - \vec{x}\,')}{|\vec{x} - \vec{x}\,'|^3}$$

   show that

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \qquad \text{when} \qquad \vec{\nabla}\cdot\vec{J} \neq 0$$

Solution

a) Charge in a volume V is given by $Q_V = \int_V d^3x\rho$. Then,

$$\frac{\partial Q_V}{\partial t} = \int_V d^3x \frac{\partial \rho}{\partial t} = -\int_V d^3x \vec{\nabla}\cdot\vec{J} = -\oint_S d\vec{S}\cdot\vec{J} = -I_S$$

where, I_S is the current flowing through the boundary S of the volume V, and we have used the divergence theorem. Thus, any change in the charge Q_V contained in volume V is entirely due to the flow of charge into or out of V, across the surface S. There is no creation or destruction of charge inside V.

b)We need the following relations,

$$\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|} = -\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = -\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}, \quad \nabla^2\frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\delta^3(\vec{x}-\vec{x}'),$$

and $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$. Note that $\vec{\nabla}$ involves differentiations with respect to $\vec{x}$ while $\vec{\nabla}'$ involves differentiations with respect to $\vec{x}'$. This difference should be kept in mind. Then $\vec{B}(\vec{x}) = \frac{1}{c}\vec{\nabla}\times\int d^3x'\vec{J}(\vec{x}')/|\vec{x}-\vec{x}'|$ and

$$\begin{aligned}
\vec{\nabla}\times\vec{B}(\vec{x}) &= \frac{1}{c}\vec{\nabla}\times\left(\vec{\nabla}\times\int d^3x'\vec{J}(\vec{x}')/|\vec{x}-\vec{x}'|\right) \\
&= \frac{1}{c}\vec{\nabla}\left(\int d^3x'\vec{J}(\vec{x}')\cdot\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}\right) - \frac{1}{c}\int d^3x'\vec{J}(\vec{x}')\nabla^2\frac{1}{|\vec{x}-\vec{x}'|} \\
&= -\frac{1}{c}\vec{\nabla}\left(\int d^3x'\vec{J}(\vec{x}')\cdot\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}\right) + \frac{4\pi}{c}\vec{J}(\vec{x}) \\
&= -\frac{1}{c}\vec{\nabla}\int d^3x'\left(\vec{\nabla}'\cdot\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} - \frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) + \frac{4\pi}{c}\vec{J}(\vec{x}) \\
&= -\frac{1}{c}\vec{\nabla}\oint d\vec{S}'\cdot\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\int d^3x'\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{4\pi}{c}\vec{J}(\vec{x})
\end{aligned} \tag{1}$$

The first integral is a surface term and vanishes. The second integral becomes,

$$-\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\int d^3x'\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\Phi(\vec{x}) = \frac{1}{c}\frac{\partial}{\partial t}\vec{E}$$

and hence the result.

3. (a) Show that the function ${G'}_k = \frac{e^{ikR}}{R}$ with $R = |\vec{x}-\vec{x}'|$ is a solution to

$$\left(\nabla^2 + k^2\right){G'}_k = -4\pi\delta^3(\vec{x}-\vec{x}')$$

   (b) Using this result, construct the spherically symmetric retarded Green function $G^+(\vec{x},t;\vec{x}',t')$ as a solution of

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G = -4\pi\delta^3(\vec{x}-\vec{x}')\delta(t-t')$$

(c) Using the retarded Green function, show that the wave equation for the scalar potential $\nabla^2\Phi - (1/c^2)(\partial^2/\partial t^2)\Phi = -4\pi\rho$ has a solution,

$$\Phi(\vec{x}, t) = \int d^3x' \left[\frac{\rho(\vec{x}', t')}{|\vec{x} - \vec{x}'|}\right]_{t' = t - \frac{\vec{x} - \vec{x}'}{c}}$$

Comment on the difference with the generalised Coulomb law of electrostatics.

Solution

a) To show this, we need

$$\vec{\nabla}R = \frac{\vec{R}}{R}, \quad \nabla^2 R = \frac{2}{R}, \quad \vec{\nabla}(\frac{1}{R}) = -\frac{\vec{R}}{R^3}, \quad \nabla^2(\frac{1}{R}) = -4\pi\delta^3(\vec{x} - \vec{x}')$$

Now, $\vec{\nabla}G'_k = ik\vec{\nabla}R\frac{e^{ikR}}{R} + \vec{\nabla}(\frac{1}{R})e^{ikR}$ *so that*

$$\begin{aligned} \nabla^2 G'_k &= \left(\frac{ik}{R}\nabla^2 R + 2ik\vec{\nabla}R\cdot\vec{\nabla}(\frac{1}{R}) + \nabla^2(\frac{1}{R}) - \frac{k^2}{R}\vec{\nabla}R\cdot\vec{\nabla}R\right)e^{ikR} \\ &= \left(-\frac{k^2}{R} - 4\pi\delta^3(\vec{x} - \vec{x}')\right)e^{ikR} \end{aligned}$$

Therefore, $(\nabla^2 + k^2)\,G'_k = -4\pi\delta^3(\vec{x} - \vec{x}')e^{ikR} = -4\pi\delta^3(\vec{x} - \vec{x}')$. *The last step follows from the fact that the delta-function is non-zero only for* $R = 0$ *and hence the exponential becomes redundant.*

b) Using the Fourier transform of $G^+(\vec{x}, t; \vec{x}', t')$ *and the representation of the time delta-function,*

$$G^+(\vec{x}, t; \vec{x}', t') = \int d\omega G_\omega\, e^{-i\omega t}, \qquad \delta(t - t') = \int d\omega\, e^{-i\omega(t-t')},$$

the equation reduces to

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)G_\omega = -4\pi\delta^3(\vec{x} - \vec{x}')e^{i\omega t'}$$

Now, with $k^2 = \omega^2/c^2$ *and* $G'_k = G_\omega e^{-i\omega t'}$ *it becomes,*

$$\left(\nabla^2 + k^2\right)G'_k = -4\pi\delta^3(\vec{x} - \vec{x}')$$

The solution of this equation was given in part a) as $G'_k = \frac{e^{ikR}}{R}$. *Using this, the Fourier transform for* G^+ *becomes,*

$$G^+(\vec{x}, t; \vec{x}', t') = \int d\omega\,\frac{e^{ikR}}{R}\,e^{i\omega(t'-t)} = \frac{1}{R}\int d\omega\, e^{i\omega(t'-t+R/c)} = \frac{1}{R}\,\delta(t' - t + R/c)$$

Thus, $G^+ \neq 0$ *for* $t = t' + R/c$

c) The equation of part b) is the Green's function equation associated with $\nabla^2\Phi - (1/c^2)(\partial^2/\partial t^2)\Phi = -4\pi\rho$. *Hence the solution for* Φ *is given by*

$$\Phi(\vec{x}, t) = \int d^3x' \int dt' G^+(\vec{x}, t; \vec{x}', t')\rho(\vec{x}', t') = \int d^3x' \left[\frac{\rho(\vec{x}', t')}{|\vec{x} - \vec{x}'|}\right]_{t' = t - \frac{\vec{x} - \vec{x}'}{c}}$$

where the solution for G^+ has been used. If ρ is time independent, then this reduces to the generalized Coulomb law of electrostatics. In that case, if we introduce a time dependence in ρ by hand, then Φ will have the same time dependence, implying that a change in ρ at x' is instantly transmitted to Φ at x giving rise to action at a distance. However the use of the retarded Green's function implies that a change in ρ propagates outward at the velocity of light and not instantaneously.

4. A straight wire of length L and radius a has a resistance R and carries current I.

   (a) Find the electric and magnetic fields on the surface of the wire and indicate their directions.

   (b) Show that the flux of the Poynting vector across the surface of the wire is I^2R. Interpret this result in terms of the Poynting theorem.

   Solution
   a)The electric field on the surface is given by $E = \Phi/L$ where the constant potential difference Φ is given by Ohm's law, $\Phi = IR$. Hence, $E = IR/L$. The direction of $\vec{E}$ is parallel to the current and hence to the wire. The magnetic field on the surface is given by Ampere's law as $B = 2I/ca$ (This is obtained by integrating $\vec{\nabla}\times\vec{B} = (4\pi/c)\vec{J}$ over a cross section of the wire and using the cylindrical symmetry of the problem). The direction of $\vec{B}$ is given by the "right-hand-rule" which makes it perpendicular to both $\vec{E}$ and the radius vector of the cylindrical wire. Hence the direction of $\vec{B}$ is along the angular direction of the cylinder.
   b) The Poynting vector is $\vec{S} = (c/4\pi)\vec{E}\times\vec{B}$. Since $\vec{E}$ is perpendicular to $\vec{B}$, we have for the magnitude of $S = I^2R/(2\pi aL)$. $\vec{S}$ is directed radially inward. The flux $\int \vec{ds}\cdot\vec{S}$ evaluated over the surface of a segment of length L of the cylindrical wire receives contributions only from the curved side-area of the cylinder (of area $2\pi aL$) and not from the top and bottom caps (since $\vec{S}$ is parallel to them). Hence the total flux is I^2R.

   Since the electric and magnetic fields are constant, the Poynting theorem reduces to the statement that the energy carried into the wire by the Poynting vector is fully converted into the kinetic energy of the charge carriers. Since the current is constant, the system is in steady state and the extra kinetic energy acquired by the charges is dissipated into heat as a result of collisions within the resistive medium. The resulting Ohmic power loss as we know is I^2R consistent with the Poynting theorem.

5. (a) Starting from the fact that the conservation of electric charge holds in all inertial frames, find the transformation properties of electric current and charge densities under Lorentz transformations.

   (b) In relativistic electrodynamics, the Lorentz force law is contained in

$$m_0\frac{dU^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}U_\nu$$

where $U^\mu = (\gamma c, \gamma\vec{u})$ is the relativistic 4-velocity, τ is time in the rest-frame of the moving charge ($dt = \gamma d\tau$) and $\gamma^{-1} = \sqrt{1 - u^2/c^2}$.

i. Show how this modifies the non-relativistic Lorentz force law, $m(d\vec{u}/dt) = q\vec{E} + (q/c)\vec{u}\times\vec{B}$.

ii. Besides this equation,what other equation is contained in the relativistic force law? Explain its physical significance.

Solution

a) The conservation law of electric charge is expressed in terms of the continuity equation $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{J} = 0$, which can be re-expressed as $\partial_\mu j^\mu = 0$, where $\partial_\mu = \{\frac{\partial}{c\partial t}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\}$ and $j^\mu = \{c\rho, j^1, j^2, j^3\}$, for $\mu = 0,1,2,3$. It is known that under Lorentz transformations, the quantity ∂_μ transforms as a covariant 4-vector. Since the conservation law must hold in all Lorentz frames, $\partial_\mu j^\mu$ must be a scalar which is true if j^μ transforms as a contravariant 4-vector. Thus under a Lorentz transformation $x'^\mu = L^\mu{}_\nu x^\nu$ one has $j'^\mu = L^\mu{}_\nu j^\nu$, or $\rho' = \frac{1}{c}L^0{}_\nu j^\nu$ and $j'^i = L^i{}_\nu j^\nu$ for $i = 1,2,3$.

b)(i) For $\mu = i$, the relativistic equation reduces to $m(du^i)/dt) + m_0 u^i(d\gamma/dt) = qE^i + (q/c)\epsilon^{ijk}u^j B_k$. where $m = \gamma m_0$. The relativistic correction is the term involving $d\gamma/dt$. Note that γ being a function of the velocity $\vec{u}$ of the moving particle, is not constant in time.

b)(ii) For $\mu = 0$, it reduces to $d(m_0c^2\gamma)/dt = qE^iu^i$. We recognize $\mathcal{E} = m_0c^2\gamma$ as the relativistic energy of the particle. Hence $d(\mathcal{E})/dt = q\vec{u}\cdot\vec{E}$ which gives the power transferred to the charged particle from the electric field.

6. An observer in a frame S measures the non-zero components of the electric and magnetic fields as $E^2(\vec{x})$ and $B^3(\vec{x})$. If a frame $\tilde{S}$ is moving with respect to S with velocity v in the positive x^1 direction, find the components $\tilde{E}^i(\vec{\tilde{x}})$ and $B^i(\vec{\tilde{x}})$ of the electric and magnetic fields as measured by the observer in $\tilde{S}$.

Solution

In the matrix notation, under Lorentz transformations, the electromagnetic fieldstrength tensor $F^{\mu\nu}$ transforms as $\widetilde{F}(\widetilde{x}) = L\,F(x)\,L^T$, where at the end x^μ in the right hand side should be expressed in terms of $\widetilde{x}^\mu$ using $x = L^{-1}\widetilde{x}$. In this case, the matrices F and L are given by

$$F = \begin{pmatrix} 0 & 0 & -E^2 & 0 \\ 0 & 0 & -B^3 & 0 \\ E^2 & B^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

leading to

$$\begin{aligned} \widetilde{E}^1 &= 0 & \widetilde{B}^1 &= 0 \\ \widetilde{E}^2 &= \gamma(E^2 - \beta B^3) & \widetilde{B}^2 &= 0 \\ \widetilde{E}^3 &= 0 & \widetilde{B}^3 &= \gamma(B^3 - \beta E^2) \end{aligned}$$

The arguments of E^2 and B^3 should now be expressed in terms of $\widetilde{x}^\mu$ using $x = L^{-1}\widetilde{x}$.