Final Examination Paper for Electrodynamics-I

Allowed help material: *Physics and Mathematics handbooks*

Date: *Tuesday, January 04, 2005*, Time: *09:00 - 15:00* **[Solutions]**

Questions:	1a	1b	1c	2a	2b	3a	3b	3c	4a	4b	5a	5b	5c	6a	6b	Total
Marks:	5	4	5	4	5	5	4	5	5	5	5	4	5	5	4	70

1. (a) For arbitrary functions ϕ and ψ, prove Green's second identity,

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3x = \oint_S (\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n})dS$$

   (b) Let ϕ satisfy the Poisson equation, $\nabla^2\phi = -4\pi\rho$. Write down the equation for the corresponding Green's function $G(\vec{x}, \vec{x}')$ and express the general solution of the Poisson equation in terms of $G(\vec{x}, \vec{x}')$.

   (c) Consider a magnetic vector potential $\vec{A}$ which is non-zero within a volume V. Show that it is always possible to find a gauge transformed potential $\vec{A}'$ that satisfies the Coulomb gauge $\nabla \cdot \vec{A}' = 0$.

Solution

a) This is worked out in section 1.8 of Jackson's book (third edition)
b) This is given in section 1.10 of Jackson's book.
c) $\vec{A}$ and $\vec{A}'$ are related by a gauge transformation, $\vec{A}' = \vec{A} + \nabla\Lambda$. In general $\vec{A}$ does not satisfy the Coulomb gauge. Demanding that $\vec{A}'$ satisfies the Coulomb gauge amounts to $\nabla^2\Lambda = -\nabla \cdot \vec{A}$. One now has to argue that for any $\vec{A}$, a solution Λ to this equation exists. For this, note that Λ satisfies the Poisson equation with $4\pi\rho = \nabla \cdot \vec{A}$ and part (b) of the question essentially tells us that one can always write down a solution in terms of an appropriate Green's function.

2. (a) Show that the equation $\nabla \times \vec{E} = 0$ implies that the electrostatic field $\vec{E}$ can be derived from a potential ϕ as $\vec{E} = -\nabla\phi$

   (b) Show that the energy density stored in an electrostatic field is given by $w = \frac{1}{8\pi}|\vec{E}|^2$ (or $w = \frac{\epsilon_0}{2}|\vec{E}|^2$ in SI units).

Solution

a) This is worked out in the lecture notes "Rapid Review of Vector Calculus" section 1.4. The strategy is as follows: Stokes's theorem leads to $\oint_l \vec{E} \cdot \vec{dl}$ for a closed path l, which shows the path independence of the integral. Then $\int_A^B \vec{E} \cdot \vec{dl} = \Phi(A, B)$, that is, the value of the integral depends only on the end points of the path. Now, by considering a small variation of the point B, $\vec{r}_B \to \vec{r}_B + \vec{\delta r_B}$ on both sides of the above equation, one concludes that $\vec{E} = -\nabla\phi$ (up to a sign).
b) This is worked out in section 1.11 of Jackson's book (or in the lecture notes: "From Coulomb's Law to Green's Functions").

3. Consider a grounded conducting sphere of radius a the centre of which coincides with the origin of the coordinate system. Place a point charge q at $\vec{y}$ outside it.

   (a) Find the value q' and the position $\vec{y}\,'$ of the image charge inside the sphere.

   (b) Evaluate the potential ϕ at any point $\vec{x}$ outside the sphere.

   (c) Evaluate the surface charge density σ induced on the surface of the sphere.

   Solution
   a),b),c) These are solved in section 2.2 of Jackson's book (or in the lecture notes, "Method of Images and the Sphere Green's Function")

4. (a) Show that the polarization charge density ρ_{pol} induced in a dielectric medium placed in an external electric field is given in terms of the polarization $\vec{P}$ by

$$\rho_{pol} = -\vec{\nabla} \cdot \vec{P}$$

   (b) Explain the physical origin of the above induced charge density.

   Solution
   a) This is worked out in Jackson's book, section 4.3: One starts with the expression for the electrostatic potential due to a dipole moment density in a volume ΔV. On a little manipulation, one sees that $-\nabla \cdot \vec{P}$ can be regarded as an effective charge density.
   b) This is explained in the lecture notes, "Electrostatics in Dielectric Media": Under the influence of an external field, positive and negative charges in a medium separate out giving rise to a polarization density $\vec{P}$. For uniform charge separation, $\vec{P}$ is constant ($\nabla \cdot \vec{P} = 0$). In this case, there is no net charge accumulation anywhere and hence no net polarization charge density. However if charge separation is not uniform, then there is a non-zero $\rho_{pol}(x)$. In this situation $\vec{P}$ also varies in space and hence $\nabla \cdot \vec{P} \neq 0$.

5. (a) Show that Maxwell's equations are consistent with the conservation of electric charge.

   (b) Starting from Maxwell's equations show that in free space electric and magnetic disturbances are waves travelling at the speed of light c.

   (c) Show that the power injected into a circuit by an electric field is given by $\int \vec{J} \cdot \vec{E} d^3x$. Verify that in steady state this reproduces the Ohmic heat loss in a "thin wire" approximation.

   Solution
   a) Start with the two Maxwell equations containing the sources ρ and $\vec{J}$. Taking a time derivative of the first equation and a divergence of the second, one gets $\partial\rho/\partial t + \nabla \cdot \vec{J} = 0$ which is a statement of charge conservation.
   b) Start with the two Maxwell equations containing $\nabla \times \vec{E}$ and $\nabla \times \vec{B}$. Evaluating time derivatives and curls of these equations it becomes obvious that each component

of $\vec{E}$ and $\vec{B}$ satifies the wave equation $(\nabla^2 - \frac{1}{c^2}\partial^2/\partial t^2)\psi = 0$.
c) The power transferred to a point charge q by an electric field $\vec{E}$ is the rate of change of its kinetic energy, $\frac{1}{2}mv^2$. Using the Lorentz force law and $\vec{v}\cdot(\vec{v}\times\vec{B}) = 0$, this becomes $q\vec{v}\cdot\vec{E}$. For charges contained in volume d^3x within a continuous charge distribution, one has $q \to \rho d^3x$. Using $\vec{J} = \rho\vec{v}$ and integrating over the volume of the current distribution, leads to the desired result. Now we restrict the current to a thin wire in steady state. For an element of length $\vec{dl}$ and cross section $\vec{ds}$ of the wire, $d^3x = \vec{dl}\cdot\vec{ds}$. In this approximation, $\vec{J}$ is parallel to $\vec{dl}$ and $\vec{E}$ does not vary appreciably over the cross section of the wire. Hence $\int \vec{J}\cdot\vec{E}d^3x = \int \vec{J}\cdot\vec{ds}\int_A^B \vec{E}\, dl = IV_{AB} = I^2R_{AB}$ which is the Ohmic heat loss. (This is worked out in the lecture notes: "Time-Dependent Fields, Maxwell Equations and Energy Considerations")

6. (a) Define contravariant and covariant vectors with respect to general coordinate transformations. Under a Lorentz transformation $\tilde{x}^\mu = L^\mu{}_\nu x^\nu$, how does $\partial_\mu = \frac{\partial}{\partial x^\mu}$ transform?
   (b) Starting from the fact that the conservation of electric charge holds in all intertial frames, find the transformation properties of electric charge and current densities under Lorentz transformations.

Solution
a) Covariant and Contravariant vectors are defined in Jackson's book, section 11.6 and in the lecture notes: "Special Relativity and Maxwell's Equations". Since $x^\nu = (L^{-1})^\nu{}_\mu\tilde{x}^\mu$, one has $\partial x^\nu/\partial\tilde{x}^\mu = (L^{-1})^\nu{}_\mu$ and hence $\tilde{\partial}_\mu = (L^{-1})^\nu{}_\mu\partial_\nu$.
b) As shown in section 11.9 of Jackson's book, $c\rho$ and $\vec{J}$ combine into a contravariant 4-vector J^μ and transform accordingly.