Final Examination Paper for Electrodynamics-I

Allowed help material: *Physics and Mathematics handbooks*

Date: *Saturday, October 30, 2004*, Time: *09:00 - 15:00*

Questions:	1a	1b	2a	2b	3a	3b	4a	4b	4c	4d	5a	5b	6a	6b	6c	Total
Marks:	3	7	4	4	5	3	4	4	4	4	5	5	10	6	2	70

1. (a) Prove Green's first identity,

$$\int_V (\phi \nabla^2 \psi + \vec{\nabla}\phi \cdot \vec{\nabla}\psi) d^3x = \oint_S \phi \frac{\partial \psi}{\partial n} dS$$

for two arbitrary functions ϕ and ψ.

(b) Show that the solutions of the Poisson equation, $\nabla^2 \phi = -4\pi\rho$, are uniquely specified by *Neumann* and *Dirichlet* boundary conditions (up to a constant).

Solution

a) Evaluate $\vec{\nabla} \cdot (\phi \vec{\nabla}\psi)$ and integrate the outcome over the volume V using the divergence theorem (worked out in Jackson, section 1.8)

b) For two possible solutions ϕ_1 and ϕ_2 satisfying the same boundary conditions, define $U = \phi_1 - \phi_2$ and show that $U = 0$ (Dirichlet) or $U = const$ (Neumann) (for the proof, see Jackson, section 1.9 or the class notes)

2. (a) Consider the multipole moments of a charge distribution $\rho(r, \theta, \phi)$,

$$q_{lm} = \int r^2 dr \int \sin\theta d\theta \int d\phi \ \rho(r, \theta, \phi) \, r^l \, Y^*_{lm}(\theta, \phi)$$

Show that q_{00} is total charge and that for a *spherically symmetric* charge distribution all moments beyond this monopole moment vanish.

(b) Consider the boundary between two media of dielectric constants ϵ_1 and ϵ_2. Let us denote the electric displacement vectors on the two sides of the boundary by $\vec{D}_1$ and $\vec{D}_2$ and the polarization densities by $\vec{P}_1$ and $\vec{P}_2$, respectively. Show that even in the absence of free charges, a polarization charge density

$$\sigma_{pol} = (\vec{P}_1 - \vec{P}_2) \cdot \hat{n}$$

develops on the boundary ($\hat{n}$ is a unit vector normal to the boundary).

Solution

a) Since $Y_{00}(\theta, \phi) = 1/\sqrt{4\pi}$, it is obvious that q_{00} determines the total charge Q as $q_{00} = Q/\sqrt{4\pi}$. This reminds us that Y_{00} is a constant, so we can insert $Y_{00}(\theta, \phi)/\sqrt{4\pi} = 1$ in the integral expression for q_{lm}. For a spherically symmetric charge distribution, $\rho(r, \theta, \phi) = \rho(r)$ and is not affected by the angular integrals. Then from the orthogonality of spherical harmonics it follows that $q_{lm} = 0$.

b) This directly follows from the fact that $\vec{D}$ accross the surface is continuous while the discontinuity in $\vec{E} \cdot \hat{n}$ is given by the surface charge density and $\vec{D} = \vec{E} + 4\pi\vec{P}$.

3. (a) Show that the force acting on a current distribution $\vec{J}$ placed in a magnetic field $\vec{B}$ is given by

$$\vec{F} = \frac{1}{c}\int_V \vec{J}(\vec{x}) \times \vec{B}(\vec{x})\, d^3x$$

(b) What is the physical significance of the equation $\vec{\nabla} \cdot \vec{B} = 0$?

Solution

a) This is the Lorentz force $d\vec{F} = \frac{1}{c}\rho\vec{v} \times \vec{B} d^3x$ *applied to a current in a volume* V

b) Absence of magnetic monopoles

4. Maxwell's equations in the absence of sources have travelling wave solutions,

$$\vec{E}(\vec{x}, t) = \vec{E}_{\rm o} e^{i(\vec{k}\cdot\vec{x}-\omega t)}\,, \qquad \vec{B}(\vec{x}, t) = \vec{B}_{\rm o} e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

Show that for such solutions,

(a) $\vec{E}$ and $\vec{B}$ are perpendicular to the wave-vector $\vec{k}$ ($\vec{k} \cdot \vec{E} = \vec{k} \cdot \vec{B} = 0$).

(b) $\vec{E}$ and $\vec{B}$ are perpendicular to each other and hence, along with $\vec{k}$, form a triplet of orthogonal vectors.

(c) What is the velocity of the wave?

(d) Show that for this solution, the energy density in the electric field is equal to the energy density in the magnetic field

[Hint: You may need the vector identity $(\vec{A} \times \vec{B}) \times \vec{C} = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{A}\,(\vec{B} \cdot \vec{C})$]

Solution

a) Since $\vec{\nabla} \cdot \vec{E} = i\vec{k} \cdot \vec{E}$ *(and similarly for* $\vec{B}$*), the Maxwell's equations for* $\vec{\nabla} \cdot \vec{E}$ *and* $\vec{\nabla} \cdot \vec{B}$ *lead to this result.*

b) Now substituting the solutions in the remaining two Maxwell's equations gives $\vec{k} \times \vec{E} = \frac{\omega}{c}\vec{B}$ *and* $\vec{k} \times \vec{B} = -\frac{\omega}{c}\vec{E}$ *which proves the point.*

c) Using the Hint, eliminate $\vec{E}$ *or* $\vec{B}$ *between the two equations in b), leading to* $c = \omega/|k| = \nu\lambda$.

d) From b) and c) it follows that $|\vec{E}| = |\vec{B}|$ *so that* $\frac{1}{8\pi}E^2 = \frac{1}{8\pi}B^2$.

5. (a) Prove the *Poynting theorem*,

$$\vec{J} \cdot \vec{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon\vec{E} \cdot \vec{E} + \frac{1}{\mu}\vec{B} \cdot \vec{B}\right) + \frac{c}{4\pi}\nabla \cdot (\vec{E} \times \vec{H}) = 0$$

[Hint: You may need the vector identity $\nabla \cdot (\vec{P} \times \vec{Q}) = (\nabla \times \vec{P}) \cdot \vec{Q} - \vec{P} \cdot (\nabla \times \vec{Q})$]

(b) Give the physical interpretation of each term in this equation. What is the significance of the *Poynting theorem*?

Solution

a) See section 6.8 in Jackson or the class notes

b) First term: power injected into the current distribution by the electric field/unit *volume, Second term: rate of change of energy density of the electric and magnetic fields, Third term: energy flux per unit time per unit volume carried by the em fields. The Poynting theorem is a statement of conservation of energy and also indicates that energy is carried by electromagnetic waves.*

6. (a) Consider a point charge q which is at rest at the origin of an inertial frame S. A frame $\widetilde{S}$ is moving away from S in the x^1 direction with constant velocity v. Find the electric and magnetic potentials $\widetilde{\phi}(\tilde{x})$ and $\widetilde{A}^i(\tilde{x})$ due to q as measured by an observer in the $\widetilde{S}$ frame.

   (b) Consider a Lorentz transformation L relating two reference frames S and $\widetilde{S}$. Then the space-time coordinates of a particle measured in the two frames are related by

$$\tilde{x}^\mu = \sum_{\nu=0}^{3} L^\mu{}_\nu \, x^\nu$$

   Assume that an observer in S measures the velocity of the particle as $\vec{u}$ (so that in time Δt it is displaced by $\Delta x^i = u^i \Delta t$). Find the particle velocity $\tilde{u}^i$ in the frame $\widetilde{S}$ in terms of $L^\mu{}_\nu$ and u^i.

   (c) When frame $\widetilde{S}$ moves away from S with velocity v in the x^1 direction, check that the above relation between $\tilde{u}^i$ and u^i reduces to the standard form of the law of addition of velocities,

$$\tilde{u}^1 = \frac{u^1 - v}{1 - \dfrac{vu^1}{c^2}}\,, \qquad \tilde{u}^2 = \frac{u^2}{\gamma\left[1 - \dfrac{vu^1}{c^2}\right]} \qquad \tilde{u}^3 = \frac{u^3}{\gamma\left[1 - \dfrac{vu^1}{c^2}\right]}$$

Solution

a) We use $\tilde{A}^\mu(\tilde{x}) = \sum_{\nu=0}^{3} L^\mu{}_\nu \, A^\nu(x)$. Since the charge is stationary in S, $\phi = q/r$, $A^i = 0$. Also for this problem, $L^0{}_0 = L^1{}_1 = \gamma$ and $L^0{}_1 = L^1{}_0 = -\gamma\beta^1$. It then follows that $\tilde{\phi}(\tilde{x}) = \gamma q/r$ and $\tilde{A}^1 = -(\gamma v)q/cr$. This is still in terms of $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ and should be expressed in terms of $\tilde{x}^i$. Under the Lorentz transformation, $x^1 = \gamma(\tilde{x}^1 + v\tilde{t})$ with x^2 and x^3 unchanged. Hence the final answer is, $\tilde{\phi}(\tilde{x}) = \gamma q/\sqrt{\gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (x^2)^2 + (x^3)^2}$ and $\tilde{A}^1 = -(\gamma v)q/c\sqrt{\gamma^2(\tilde{x}^1 + v\tilde{t})^2 + (x^2)^2 + (x^3)^2}$.
b) Note that $\tilde{u}^i/c = \Delta\tilde{x}^i/\Delta\tilde{x}^0$ $(i = 1, 2, 3)$, where $\Delta\tilde{x}^\mu = \sum_{\nu=0}^{3} L^\mu{}_\nu \, \Delta x^\nu$. Now, using the fact that $\Delta x^i = u^i \Delta t$ and $\Delta x^0 = c\Delta t$, we get

$$\frac{\tilde{u}^i}{c} = \frac{L^i{}_j u^j + L^i{}_0 c}{L^0{}_j u^j + L^0{}_0 c}$$

c) Using the form of $L^\mu{}_\nu$ as given in the solution to part a) of the problem in the result in part b) we obtain the desired result.