Solutions to the Maxwell Equations

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(Class notes – S. F. Hassan, 2006)

1 The Wave Equations

Maxwell's equations in cgs units are

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \tag{1}$$

$$\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J} \tag{2}$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \tag{3}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{4}$$

with

$$\vec{D} = \epsilon\vec{E} \qquad \vec{H} = \frac{1}{\mu}\vec{B}$$

For the present, ϵ and μ are taken to be real constants corresponding to homogeneous non-conducting media. In the cgs system in free space, $\mu = 1$, $\epsilon = 1$. Taking the curl of the second and the third equation using the identity

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{E} + \vec{\nabla}(\vec{\nabla}\cdot\vec{E})$$

along with the first and the fourth equation, we end up with decoupled second order equatios for $\vec{E}$ and $\vec{B}$,

$$\nabla^2\vec{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \frac{4\pi\mu}{c^2}\frac{\partial\vec{J}}{\partial t} + \frac{4\pi}{\epsilon}\vec{\nabla}\rho\,, \tag{5}$$

$$\nabla^2\vec{B} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = -\frac{4\pi\mu}{c}\vec{\nabla}\times\vec{J} \tag{6}$$

In the absence of source terms, these reduce to the simple wave equations,

$$\nabla^2\vec{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0\,, \qquad \nabla^2\vec{B} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0 \tag{7}$$

The most general solution of these equations is of the form

$$\vec{E}(\vec{x},t) = \int d^3k \int d\omega \vec{E}(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{x}-\omega t)}\delta(k^2 - \frac{\omega^2}{c^2}\mu\epsilon) \tag{8}$$

2 Plane Wave Solutions

The simplest solutions to these equations are the so called plane wave solutions,

$$\vec{E}(\vec{x},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\,, \qquad \vec{B}(\vec{x},t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\,, \tag{9}$$

with

$$k^2 = \frac{\mu\epsilon}{c^2}\omega^2\,, \quad \text{or} \quad \omega = \pm\frac{\mathrm{c}}{\sqrt{\mu\epsilon}}|\vec{\mathrm{k}}|$$

To verify this, note that $\partial\vec{E}/\partial t = -i\omega\vec{E}$ and $\partial\vec{E}/\partial x^j = ik_j\vec{E}$ so that,

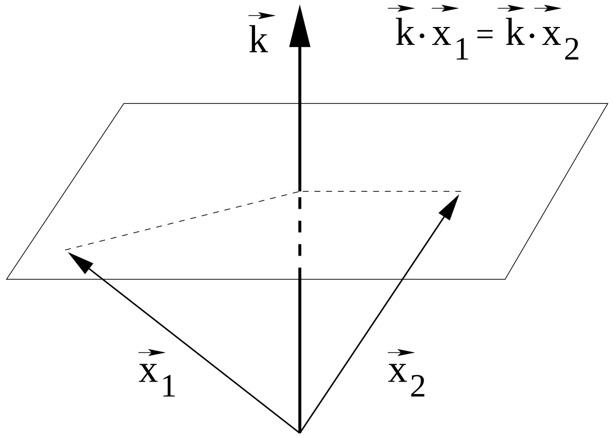
$$\vec{\nabla}\cdot\vec{E} = i\vec{k}\cdot\vec{E}\,, \quad \vec{\nabla}\times\vec{E} = i\vec{k}\times\vec{E} \quad \nabla^2\vec{E} = -k^2\vec{E}$$

with similar expressions for $\vec{B}$ (henceforth we only discuss $\vec{E}$, as the extension to $\vec{B}$ is obvios). Let us now discuss some features of the solution:

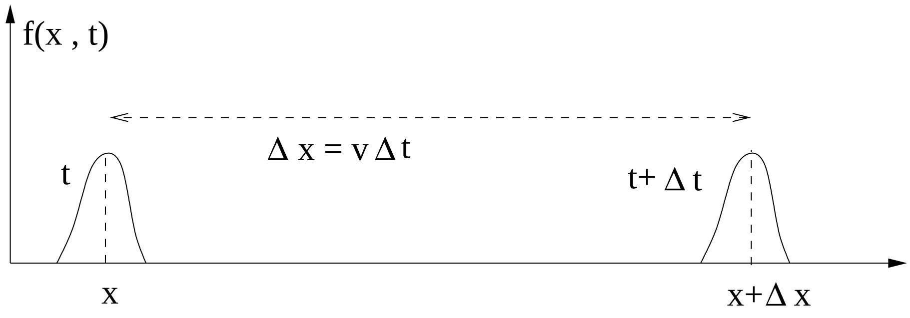
Reality: Although the physical electric field is real, mathematically it is more convenient to work with the above complex expressions. The physical electric field is then identified as the real part of $\vec{E}$. So for $\vec{E}_0 = \vec{A} - i\vec{B}$, with $\vec{A}$ and $\vec{B}$ real,

$$\left(\vec{E}(\vec{x},t)\right)_{physical} = \boldsymbol{Re}(\vec{E}) = \vec{A}\cos(\vec{k}\cdot\vec{x}-\omega t) + \vec{B}\sin(\vec{k}\cdot\vec{x}-\omega t)$$

Plane Wave Nature: The space and time dependence of the electric field is entirely contained in the phase $\varphi = \vec{k}\cdot\vec{x} - \omega t$. At a given time t, the electric field has the same value at all points $\vec{x}$ for which $\vec{k}\cdot\vec{x}$ remains fixed, *i.e.*, for all position vectors $\vec{x}$ that have the same projection along $\vec{k}$. As is obvious from the figure, all such points lie on a plane perpendicular to $\vec{k}$, hence the name plane wave. As time increases ($t \to t + \Delta t$), the plane corresponding to a given value of φ, and hence a given value of $\vec{E}$, moves along $\vec{k}$ ($\vec{x} \to \vec{x} + \vec{\Delta x}$) with velocity $v = \hat{k}\cdot\vec{\Delta x}/\Delta t = \omega/k$.



Velocity of the Wave: The velocity of the wave can be easily read off as follows: Consider a pulse $f(x,t)$ moving in the x-direction with velocity v. Then at time $t+\Delta t$, the pulse moves by $\Delta x = v\Delta t$ but its shape remains unchanged. Mathematically this is expressed as $f(x+\Delta x, t+\Delta t) = f(x,t)$ which relates the x and t dependences of f as $f(x,t) = f(x-vt)$.



This can be compared with the plane wave with phase $\varphi = k(\hat{k} \cdot \vec{x}) - \omega t$. Demanding

$$k\left(\hat{k} \cdot (\vec{x} + \vec{\Delta x})\right) - \omega(t + \Delta t) = k \cdot (\hat{k}\vec{x}) - \omega t$$

we get the velocity in the direction of $\vec{k}$ (also called the phase velocity) as

$$v = \frac{\hat{k} \cdot \Delta x}{\Delta t} = \frac{w}{k} = \frac{c}{\sqrt{\mu\epsilon}}$$

In free space, $v = c$. The observation that electromagnetic waves propagate with the velocity light led Maxwell to the conclusion that light is an electromagnetic wave. In medium the velocity is lower than c and in general one can define the index of refraction,

$$n = \frac{c}{v} = \sqrt{\mu\epsilon} \qquad (cgs)$$

Here we only consider wave propagation in non-dispersive media. In a dispersive medium, the wave velocity v will also depend on the wave frequency ω.

Transverse Nature: The plane wave solutions for $\vec{E}$ and $\vec{B}$ must also satisfy the Maxwell equations in the absence of sources. One can then see that $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{B} = 0$ imply

$$\vec{k} \cdot \vec{E} = 0\,, \qquad \vec{k} \cdot \vec{B} = 0$$

So the vectors $\vec{E}$ and $\vec{B}$ are perpendicular to $\vec{k}$. Furthermore, the remaining two equations $\vec{\nabla}\times\vec{E} + (1/c)\partial\vec{B}/\partial t = 0$ and $\vec{\nabla}\times\vec{B} - (\mu\epsilon/c)\partial\vec{E}/\partial t = 0$ imply

$$\vec{B} = \frac{c}{\omega}\,(\vec{k} \times \vec{E})\,, \qquad \vec{E} = -\frac{c}{\omega}(\frac{1}{\mu\epsilon})\,(\vec{k} \times \vec{B})$$

So the oscillating electric and magnetic fields are perpendicular to each other as well as to $\vec{k}$. This is an important property of electromagnetic plane waves. Since $|\vec{k} \times \vec{E}| = |\vec{k}||\vec{E}|$, we have

$$|\vec{B}| = \sqrt{\mu\epsilon}\,|\vec{E}|$$

In particular, for a plane wave in free space the electric and magnetic fields have the same value (in cgs units).

The angular frequency ω and wavenumber k are related to the frequency ν and wavelength λ of the oscillations through $\omega = 2\pi\nu$ and $\lambda = 2\pi/k$. The "monochromatic" plane waves described here can be linearly combined to to produce a general solution as in (8).

3 Green Functions for the Wave Equation

We have seen that Maxwell equations lead to the following wave equations for the electric and magnetic fields (from now on we consider free space, where $\epsilon = 1$, $\mu = 1$):

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = (\frac{4\pi}{c^2}\frac{\partial\vec{J}}{\partial t} + 4\pi\vec{\nabla}\rho)\,, \qquad \nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = -\frac{4\pi}{c}\vec{\nabla}\times\vec{J} \qquad (10)$$

Equivalently, one can work with the wave equations for the electric and magnatic potentials which, in the Lorenz gauge, are given by

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho\,, \qquad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J} \tag{11}$$

In the absence of sources the simplest solutions to these equations are the plane waves discussed above. We now describe the solutions in the presence of sources with the help of Green functions.

Let us consider the generic wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\vec{x},t) = -4\pi f(\vec{x},t) \tag{12}$$

The corresponding Green function $G(\vec{x},t;\vec{x}',t')$ satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G = -4\pi\delta^3(\vec{x}-\vec{x}')\delta(t-t') \tag{13}$$

As can be easily verified, in terms of G, the solution of the wave equation can be written as

$$\psi(\vec{x},t) = \psi_0(\vec{x},t) + \int d^3x' \int dt' G(\vec{x},t;\vec{x}',t')f(\vec{x}',t') \tag{14}$$

where ψ_0 is the solution of the homogeneous equation. Our purpose here is to determine the Green function G in spherically symmetric situations.

We first write equation (13) in terms of the Fourier transforms of G and the time delta-function,

$$G(t) = \int d\omega G_\omega\, e^{-i\omega t}\,, \qquad \delta(t-t') = \int d\omega\, e^{-i\omega(t-t')}$$

Comparing the coefficents of $e^{-i\omega t}$ on both sides, we get

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) G_\omega = -4\pi\delta^3(\vec{x}-\vec{x}')e^{i\omega t'}$$

which takes a simpler form in terms of $k^2 = \omega^2/c^2$ and ${G'}_k = G_\omega e^{-i\omega t'}$,

$$\left(\nabla^2 + k^2\right) {G'}_k = -4\pi\delta^3(\vec{x}-\vec{x}') \tag{15}$$

Let us now consider spherically symmetric situations where G depends only on $R = |\vec{x}-\vec{x}'|$. In this case the solutions can be written as

$$G'^{\pm}_k = \frac{e^{\pm ikR}}{R} \tag{16}$$

To verify that this solves (15), first consider the case $R \neq 0$. We use $\vec{R} = \vec{x} - \vec{x}'$ so that $\partial/\partial x^i = \partial/\partial R^i$ and hence $\vec{\nabla}_{\vec{x}} = \vec{\nabla}_{\vec{R}}$. We also have

$$\vec{\nabla}R = \frac{\vec{R}}{R}\,, \quad \nabla^2 R = \frac{2}{R}\,, \quad \vec{\nabla}(\frac{1}{R}) = -\frac{\vec{R}}{R^3}\,, \quad \nabla^2(\frac{1}{R}) = 0 \ (for\, R \neq 0)$$

It is then easy to verify that

$$
\begin{aligned}
\nabla^2 G'^{\pm}_k &= \left(\pm\frac{ik}{R}\nabla^2 R \pm 2ik\vec{\nabla}R\cdot\vec{\nabla}(\frac{1}{R}) + \nabla^2(\frac{1}{R}) - \frac{k^2}{R}\vec{\nabla}R\cdot\vec{\nabla}R \right) e^{\pm ikR} \\
&= -k^2\left(\frac{e^{\pm ikR}}{R}\right) = -k^2\, G'^{\pm}_k
\end{aligned}
$$

so equation (15) is satisfied for $R \neq 0$. To verify the presence of the delta-function source term at $R = 0$ (where the left hand side can no longer be manipulated as above due to the singular behaviour of $G'^{\pm}_k$) we use the standard technique of integrating both sides of (15) over the volume V of a sphere of some radius a centered at $\vec{R} = 0$. The right hand side gives -4π. As for the left hand side, using the divergence theorem and the spherical symmetry of the problem, one has

$$
\begin{aligned}
\int_V d^3R\,\vec{\nabla}\cdot\vec{\nabla}G'^{\pm}_k &= \oint_S \vec{ds}\cdot\vec{\nabla}(\frac{e^{\pm iRk}}{R})\Big|_{R=a} = \oint_S ds\frac{\partial}{\partial R}(\frac{e^{\pm iRk}}{R})\Big|_{R=a} \\
&= 4\pi a^2\left(\pm\frac{ik}{a} - \frac{1}{a^2}\right)e^{\pm ika}
\end{aligned}
$$

where in the third step we have used the fact that both $\vec{\nabla}G'_k$ and $\vec{ds}$ are along $\hat{R}$, resulting in the corresponding directional derivative. Also,

$$
\begin{aligned}
\int_V d^3G'^{\pm}_k &= 4\pi\int_0^a dR\,Re^{\pm ikR} = \mp 4\pi i\frac{\partial}{\partial k}\left(\int_0^a dR\,e^{\pm ikR}\right) \\
&= 4\pi\left(\frac{1}{k^2} \mp \frac{ia}{k}\right)e^{\pm ika} - \frac{4\pi}{k^2}
\end{aligned}
$$

Combining these, one has

$$
LHS = \int_V d^3R\,(\nabla^2 + k^2)\,G'^{\pm}_k = -4\pi = RHS
$$

which completes the verification of the solution (16)

Using this, we can now construct the solutions $G^{\pm}(\vec{x},t;\vec{x}',t')$ of (13),

$$
\begin{aligned}
G^{\pm}(\vec{x},t;\vec{x}',t') &= \int d\omega\,\frac{e^{\pm ikR}}{R}\,e^{i\omega(t'-t)} = \frac{1}{R}\int d\omega\,e^{i\omega(t'-t\,\pm\,R/c)} \\
&= \frac{1}{R}\,\delta(t'-t\pm R/c) \qquad (17)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&G^+ \neq 0 \quad \text{for} \quad t = t' + R/c \quad \text{(retarded Green function)} \\
&G^- \neq 0 \quad \text{for} \quad t = t' - R/c \quad \text{(advanced Green function)}
\end{aligned}
$$

G^+ has the interpretation of a wave emitted at $t', \vec{x}'$ that is detected at $\vec{x}$ at the later time t. The wave takes an amount of time $R/c = |\vec{x} - \vec{x}'|/c$ to reach from $\vec{x}'$ to $\vec{x}$ and hence corresponds to a signal travelling forward in time. On the other hand, using the same conventions, G^- has the naive interpretation of a signal that moves backwards in time (it is detected at time t before it is created at a later time t'). However, this is a *naive* interpretation and one may also regard this as signal travelling from (x, t) to (x', t'), where it is detected. Hence the advanced Green function could also occure in the description of physical processes.

It is instructive to compare $G^\pm$ (17)with the Green function for the Poisson equation that was encountered in our study of electrostatics. In that case when the boundary is at infinity, the Green function simply becomes $1/R$.

Having obtained the Green functions, one can now construct the solutions ψ of the wave equation using (14). For example, using the retarded function, one gets

$$\begin{aligned} \psi(\vec{x}, t) &= \psi_0(\vec{x}, t) + \int d^3x' \int dt' \frac{\delta(t' - t + R/c)}{|\vec{x} - \vec{x}'|} f(\vec{x}', t') \\ &= \psi_0(\vec{x}, t) + \int d^3x' \int dt' \int d^3x' \left[\frac{f(\vec{x}', t')}{|\vec{x} - \vec{x}'|}\right]_{ret} \end{aligned} \tag{18}$$

where we use the notation $[f(\vec{x}', t')]_{ret} = [f(\vec{x}', t')]_{t'=t-\frac{\vec{x}-\vec{x}'}{c}}$

As an example, let us consider the wave equation for the scalar potential, $\nabla^2\Phi - (1/c^2)(\partial^2/\partial t^2)\Phi = -4\pi\rho$ which has the solution,

$$\Phi(\vec{x}, t) = \int d^3x' \left[\frac{\rho(\vec{x}', t')}{|\vec{x} - \vec{x}'|}\right]_{t' = t - \dfrac{\vec{x} - \vec{x}'}{c}} \tag{19}$$

We have dropped ψ_0 since it is the contribution to the potential not coming from ρ. If we suppress the time dependence in the above, then we recover the generalized Coulomb law of electrostatics. Introducing a trivial time dependence in the generalized Coulomb law (so that Φ and ρ both depend on the same time t) is certainly inconsistent as it would mean that changes in ρ at $\vec{x}'$ cause an instantaneous change in Φ at $\vec{x}$ in contradiction with the fact that information cannot travel faster than light. However, the appearance of the retarded time in ρ overcomes this problem (strictly speaking, the observables are $\vec{E}$ and $\vec{B}$ and not Φ, but in any case, the above statement is true in the Lorenz gauge). Note that although G^+ was spherically symmetric, there is no such restriction on ρ and hence on Φ. The spherical symmetry of $G^\pm$ simply refers to the geometry of the surface on which we would like to impose spatial boundary conditions, in this case the sphere at infinity.

Similarly for the vector potential one gets,

$$\vec{A}(\vec{x}, t) = \frac{1}{c} \int d^3x' \frac{\left[\vec{J}(\vec{x}', t')\right]_{ret}}{|\vec{x} - \vec{x}'|} \tag{20}$$

where we have used the notation introduced below (18)

4 Jefimenko's Generalizations of the Coulomb and Biot-Savart Laws

It is now straightforward to write solutions to the electric and magnetic field equations (10) using the retarded Green function:

$$\vec{E}(\vec{x},t) = -\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\left[\frac{1}{c^2}\frac{\partial \vec{J}(\vec{x}',t')}{\partial t'} + \vec{\nabla}'\rho(\vec{x}',t')\right]_{ret} \tag{21}$$

$$\vec{B}(\vec{x},t) = \frac{1}{c}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\left[\vec{\nabla}'\times\vec{J}(\vec{x}',t')\right]_{ret} \tag{22}$$

These can be rewritten by noting that in $[f(\vec{x}',t')]_{ret}$, the variable $t' = t - (|\vec{x}-\vec{x}'|)/c$, is a function of $\vec{x}'$ and $\vec{x}$ so that,

$$\begin{aligned}\frac{\partial}{\partial x'^i}\left[f(\vec{x}',t')\right]_{ret} &= \left[\frac{\partial f(\vec{x}',t')}{\partial x'^i}\right]_{ret} + \left[\frac{\partial f(\vec{x}',t')}{\partial t'}\right]_{ret}\frac{\partial t'}{\partial x'^i} \\ &= \left[\frac{\partial f(\vec{x}',t')}{\partial x'^i}\right]_{ret} + \left[\frac{\partial f(\vec{x}',t')}{\partial t'}\right]_{ret}\frac{1}{c}\frac{\partial R}{\partial x^i}\end{aligned} \tag{23}$$

where we have used $\partial R/\partial x'^i = -\partial R/\partial x^i$. Thus, it follows that

$$\vec{\nabla}'\left[f(\vec{x}',t')\right]_{ret} = \left[\vec{\nabla}'f(\vec{x}',t')\right]_{ret} + \left[\frac{\partial f(\vec{x}',t')}{\partial t'}\right]_{ret}\frac{\vec{\nabla}R}{c}$$

Since $\vec{\nabla}R = \hat{R}$ (where the "ˆ" denotes a unit vector), for the charge density we have

$$\left[\vec{\nabla}'\rho(\vec{x}',t')\right]_{ret} = \vec{\nabla}'\left[\rho(\vec{x}',t')\right]_{ret} - \left[\frac{\partial \rho(\vec{x}',t')}{\partial t'}\right]_{ret}\frac{\hat{R}}{c}$$

Similarly, using

$$\vec{\nabla}'\times\left[\vec{J}\right]_{ret} = \sum_{ijk}\epsilon^{ijk}\hat{\mathbf{1}}_i\frac{\partial}{\partial x'^j}\left[J_k\right]_{ret}$$

we have,

$$\left[\vec{\nabla}'\times\vec{J}(\vec{x}',t')\right]_{ret} = \vec{\nabla}'\times\left[\vec{J}(\vec{x}',t')\right]_{ret} + \frac{1}{c}\left[\frac{\partial \vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret}\times\hat{R}$$

These can be used to rewrite the expressions for $\vec{E}$ and $\vec{B}$. Integrating by parts and dropping a surface term at infinity, this gives,

$$\vec{E}(\vec{x},t) = \int d^3x' \left(\frac{\hat{R}}{R^2}\left[\rho(\vec{x}',t')\right]_{ret} + \frac{\hat{R}}{Rc}\left[\frac{\partial \rho(\vec{x}',t')}{\partial t'}\right]_{ret} - \frac{1}{Rc^2}\left[\frac{\partial \vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret}\right) \tag{24}$$

$$\vec{B}(\vec{x},t) = \frac{1}{c}\int d^3x' \left(\left[\vec{J}(\vec{x}',t')\right]_{ret}\times\frac{\vec{R}}{R^3} + \left[\frac{\partial \vec{J}(\vec{x}',t')}{\partial t'}\right]_{ret}\times\frac{\hat{R}}{Rc}\right) \tag{25}$$

Note that for $\partial\vec{J}/\partial t = 0$ and $\partial\rho/\partial t = 0$ we recover the Coulomb and Biot-Savart laws.

The above expressions for $\vec{E}$ and $\vec{B}$ are the basic equations that describe the generation of electric and magnetic fields, including the electromagnetic radiation, by time dependent sources and their subsequent propagation in space. Clearly, to evaluate the fields, first we have to know the time and space dependence of the sources ρ and $\vec{J}$. Secondly, we should be able to evaluate the above integrals. This is often a very difficult job as the integrands also depend on the observation point $\vec{x}$ besides $\vec{x}'$. We encountered a similar problem in electrostatics but there the method of multipole expansion allowed us to disentangle the $\vec{x}$ and $\vec{x}'$ dependences in the problem and, in particular, expand the solution in terms of integrals that only involved $\vec{x}'$. A similar, but more involved, multipole expansion can be applied to the time dependent case although it will not be discussed here.

5 Localized Oscillating Sources and Radiation

As an example consider sources with a simple sinusoidal time dependence,

$$\rho(\vec{x},t) = \rho(\vec{x})e^{-i\omega t}\,, \qquad \vec{J}(\vec{x},t) = \vec{J}(\vec{x})e^{-i\omega t}$$

Instead plugging these directly in the expressions for $\vec{E}$ and $\vec{B}$, one can take the simpler approach of first computing $\vec{A}$,

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \frac{\left[\vec{J}(\vec{x}')e^{-i\omega t'}\right]_{ret}}{|\vec{x}-\vec{x}'|} = \left(\frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}')e^{i\omega|\vec{x}-\vec{x}'|/c}}{|\vec{x}-\vec{x}'|}\right)e^{-i\omega t} \tag{26}$$

Thus, using the notation $k=\omega/c$, one has $\vec{A}(\vec{x},t) = \vec{A}(\vec{x})e^{-i\omega t}$ where

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}')e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \tag{27}$$

The magnetic field is then given by

$$\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}(\vec{x},t) = \left(\vec{\nabla}\times\vec{A}(\vec{x})\right)e^{-i\omega t}$$

To evaluate the electric field, note that from Maxwell's equations,

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = i\frac{\omega}{c}\left(\vec{\nabla}\times\vec{A}(\vec{x})\right)e^{-i\omega t}$$

so that the time dependence of $\vec{E}$ is also given by $e^{-i\omega t}$ and in particular, $\frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E}$. Now, outside the source, $\vec{J}=0$ and

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = -i\frac{\omega}{c}\vec{E}$$

Hence,

$$\vec{E} = \frac{i}{k}\vec{\nabla}\times\vec{B} \tag{28}$$

Using the above expressions, $\vec{E}$ and $\vec{B}$ can be easily computed outside the source provided $\vec{A}(\vec{x})$ given by (27) is known. But for the reasons mentioned earlier, the computation of $\vec{A}(\vec{x})$ is the difficult part of the problem!

To disentangle the $\vec{x}$ and $\vec{x}'$ dependences in (27), one should systematically carry out a multipole expansion of the integrand. Although we will not do this here, it is worthwhile to briefly describe the basic idea (for details see Jackson's book): Remember that in electrostatics the expression for the fields contained the factor $1/|\vec{x}-\vec{x}'|$ and the multipole expansion essentially involved expanding this in powers of $|\vec{x}'|/|\vec{x}| = r'/r$ mutiplied by spherical harmonics $Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$. Now, the $\vec{A}(\vec{x})$ in (27) depends on $|\vec{x}-\vec{x}'|$ also through an extra exponential factor $e^{ik|\vec{x}-\vec{x}'|}$ which should also be expanded in an appropriate manner.

We study equation (27) in another approximation scheme that is less powerful, but that brings out the physics of the problem more clearly and gives a better insight into the nature of the solution. To understand this approximation scheme, note that the problem involves 3 length scales (for convenience, let us choose the origin or the coordinate system to be located within the source):

- In terms of the angular frequency of the source, we can define a "wavelength" $\lambda = 2\pi c/\omega = 2\pi/k$. If the solution is written in terms of the retarded Green function, then λ is the distance that a disturbance travels during the time of one oscillation, moving at the speed of light. Hence the term "wavelength".
- The size of the source can be roughly characterized in terms of its linear dimension d. This enters (27) through the size of the integration region (for example, d^3 for a cubic source). Then, $|\vec{x}'| \lesssim d$.
- The location of the observation point introduces another length scale $|\vec{x}| = r$

Our main assumption is that the dimensions of the source are much smaller than the "wavelength" λ determined by its oscillation frequency, $d << \lambda$. We also assume that the distance to the observer $|\vec{x}| = r$ is much larger than the size of the source, $d << r$. Now, three situations can arise depending on how r compares with λ:

1. Near (static) zone: $d << r << \lambda$
2. Intermediate (induction) zone: $d << r \sim r$
3. Far (radiation) zone: $d << \lambda << r$

The Near Zone: In this case, $\lambda >> r$ or $kr << 1$. But since $d << r$, *i.e.*, $|\vec{x}'| << |\vec{x}|$ we also have $k|\vec{x}-\vec{x}'| << 1$ or $e^{k|\vec{x}-\vec{x}'|} \sim 1$. The solution then becomes,

$$\lim_{kr\to 0} \vec{A}(\vec{x},t) = \left(\frac{1}{c} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right) e^{-i\omega t} \tag{29}$$

This has the same form as the static case (the Biot-Savart law) except for the overall time modulation. Thus in the near region the fields have essentially a static character and the space-dependent part of the above expression admits a multipole expansion just as in the static case.

The Far Zone: In this zone, $r >> d$ or $|\vec{x}| >> |\vec{x}'|$. If the unit vector along $\vec{x}$ is denoted by $\hat{n}$, then

$$|\vec{x} - \vec{x}'| = r|\hat{n} - \frac{\vec{x}'}{r}| = r\left(1 - 2\frac{\hat{n}\cdot\vec{x}'}{r} + \frac{x'^2}{r^2}\right)^{\frac{1}{2}} \approx r - \hat{n}\cdot\vec{x}'$$

We make this approximation in the exponent. Also we let $1/|\vec{x} - \vec{x}'| \to 1/r$. Then (27) becomes,

$$\lim_{kr\to\infty} \vec{A}(\vec{x},t) = \frac{1}{c}\frac{e^{i(kr-\omega t)}}{r}\int d^3x'\,\vec{J}(\vec{x}')e^{-ik\,\hat{n}\cdot\vec{x}'} \tag{30}$$

The integral depends only on the direction of $\vec{x}$. The prefactor represents an outgoing travelling wave with a spherical wavefront and with speed ω/k. Hence it has the character of radiation. A multipole expansion of this can be obtained noting that $d << \lambda$ implies $k\vec{x}'\cdot\hat{n} << 1$. Hence the exponential can be expanded in a power series,

$$\lim_{kr\to\infty} \vec{A}(\vec{x},t) = \frac{1}{c}\frac{e^{i(kr-\omega t)}}{r}\sum_{s=0}^{\infty}\int d^3x'\,\vec{J}(\vec{x}')\frac{(-ik\,\hat{n}\cdot\vec{x}')^s}{s!} \tag{31}$$

with terms with higher s getting smaller. The largest term in this expansion is

$$(\lim_{kr\to\infty} \vec{A}(\vec{x},t))_{s=0} = \frac{1}{c}\frac{e^{i(kr-\omega t)}}{r}\int d^3x'\,\vec{J}(\vec{x}') \tag{32}$$

Intermediate Zone: In this case non of the above approximations are valid and one has to resort to the multipole expansion of the original expression (27).

6 Electric Dipole Fields and Radiation

The $s = 0$ term (32) contains

$$\vec{A}(\vec{x}) = \frac{1}{c}\frac{e^{ikr}}{r}\int d^3x'\,\vec{J}(\vec{x}') \tag{33}$$

It turns out that this term arises not only in the far zone approximation, but that it is the $l = 0$ term in the multipole expansion of the exact expression (27), so it is valid and is the dominant contribution in all three zones. To put this in a more manageable form, note that for a $J(\vec{x})$ that vanishes sufficiently rapidly on the boundary one has

$$0 = \int d^3x'\frac{\partial}{\partial x'^i}(\vec{x}'J^i(\vec{x}')) = \int d^3x'\left(\mathbf{1}^k\delta^i_k J_i + \vec{x}'(\vec{\nabla}'\cdot\vec{J})\right)$$

Using the continuity equation $\vec{\nabla}' \cdot \vec{J}(\vec{x}', t') = -\partial\rho(\vec{x}', t')/\partial t'$ and the sinusoidal time dependence of $\rho(\vec{x}', t')$, this gives,

$$\int d^3x' \vec{J} = -\int d^3x' \vec{x}' (\vec{\nabla}' \cdot \vec{J}) = -i\omega \int d^3x' (\vec{x}' \rho) = -i\omega \vec{p}$$

where we have used the definition of the dipole moment $\vec{p}$ of an electric charge distribution. Therefore,

$$\vec{A}(\vec{x}) = -\frac{i\omega}{c}\, \vec{p}\, \frac{e^{ikr}}{r} \tag{34}$$

The magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ can now be computed as (inserting back the time dependence),

$$\vec{B}(\vec{x}, t) = k^2 (\hat{n} \times \vec{p})\, \frac{e^{i(kr-\omega t)}}{r} \left(1 - \frac{1}{ikr}\right) \tag{35}$$

where the first term comes from $\nabla \times$ acting on the exponential and second term comes from it acting on $1/r$. The electric field is $\vec{E} = (i/k)\vec{\nabla} \times \vec{B}$, or

$$\vec{E}(\vec{x}, t) = k^2 (\hat{n} \times \vec{p}) \times \hat{n}\, \frac{e^{i(kr-\omega t)}}{r} + [\, 3\hat{n}(\hat{n} \cdot \vec{p}) - \vec{p}\,] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)} \tag{36}$$

It is instructive to focus on the behaviour of these fields in the near and far zone approximations. Note that the length scales appearing in these expressions are λ and r (the scale d has disappeared and is now hidden in $\vec{p}$).

In the near (static) zone, $r << \lambda$ implying $kr << 1$. Then

$$\vec{B}(\vec{x}, t) = ik(\hat{n} \times \vec{p})\, \frac{e^{-i\omega t}}{r^2} \tag{37}$$

$$\vec{E}(\vec{x}, t) = [\, 3\hat{n}(\hat{n} \cdot \vec{p}) - \vec{p}\,] \frac{e^{-i\omega t}}{r^3} \tag{38}$$

Since $|B|/|E| \sim kr << 1$, the field is mainly electric in nature and the electric component is the same as that of a static dipole field with an overall time modulation.

In far (radiation) zone, $r >> \lambda$ implying $kr >> 1$. Then,

$$\vec{B}(\vec{x}, t) = k^2 (\hat{n} \times \vec{p})\, \frac{e^{i(kr-\omega t)}}{r} \tag{39}$$

$$\vec{E}(\vec{x}, t) = k^2 (\hat{n} \times \vec{p}) \times \hat{n}\, \frac{e^{i(kr-\omega t)}}{r} = \vec{B} \times \hat{n} \tag{40}$$

Now, $|B|/|E| \sim 1$ and both components are equally strong, as in plane waves. $\vec{E}$ and $\vec{B}$ are perpendicular to each other and the solution represents expanding spherical wave-fronts.

The change in the nature of the solution from static-like to radiation-like as one gose from the near zone to the far zone can be understood qualitatively: The oscillating electric dipole produces an oscillating dipole electric field in its neighbourhood. The magnetic field in this region is insignificant. However, Maxwell's equations couple electric and magnetic fields and as they propagate the wave, the magnetic field grows by taking energy for the electric field, until the two have the same energy.

7 Boundary Conditions at Interfaces Between Different Media

Summary: During our study of electrostatics and magnetostatics we considered surface distributions of electric charge and electric current. We then saw how the normal and tangential components of electric and magnetic fields changed across this surface. Here we learn that this result does not change in the presence of time dependent fields. The derivation proceeds as in the static case and the new terms in the Maxwell's equations give zero contribution. For details see Jackson's book, section I.5 (pages 16-19 in the 3rd edition).

8 Reflection and Refraction of Electromagnetic Waves (kinematic aspects)

Consider the boundary between two media with different electric and magnetic properties specified by (ϵ, μ) and (ϵ', μ'). A plane wave of wave vector $\vec{k}$ in the (ϵ, μ) medium is incident on the boundary. We assume that it partly gets reflected back into the same medium and partly refracted in the (ϵ', μ') medium and that both these are described by plane waves of wave vectors $\vec{k}''$ and $\vec{k}'$ respectively. So we have

$$\begin{aligned} \text{Incident wave :} \quad & \vec{E} = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} \\ \text{Reflected wave :} \quad & \vec{E}'' = \vec{E}_0''\, e^{i(\vec{k}''\cdot\vec{x}-\omega'' t)} \\ \text{Refracted wave :} \quad & \vec{E}' = \vec{E}_0'\, e^{i(\vec{k}'\cdot\vec{x}-\omega' t)} \end{aligned}$$

Being plane waves,

$$k = \sqrt{\epsilon\mu}\omega/c \qquad k'' = \sqrt{\epsilon\mu}\omega''/c \qquad k' = \sqrt{\epsilon'\mu'}\omega'/c$$

where, $k = |\vec{k}|$, etc. **For details of the discussion see Jackson's book, section 7.3 (pages 302-304 in the 3rd edition)**. We will not repeat that discussion here but make some complementary remarks: At the boundary, the normal and tangential components of $\vec{E}$, $\vec{E}'$ and $\vec{E}''$ should satisfy appropriate boundary conditions. For these boundary conditions to be valid at all times, we need,

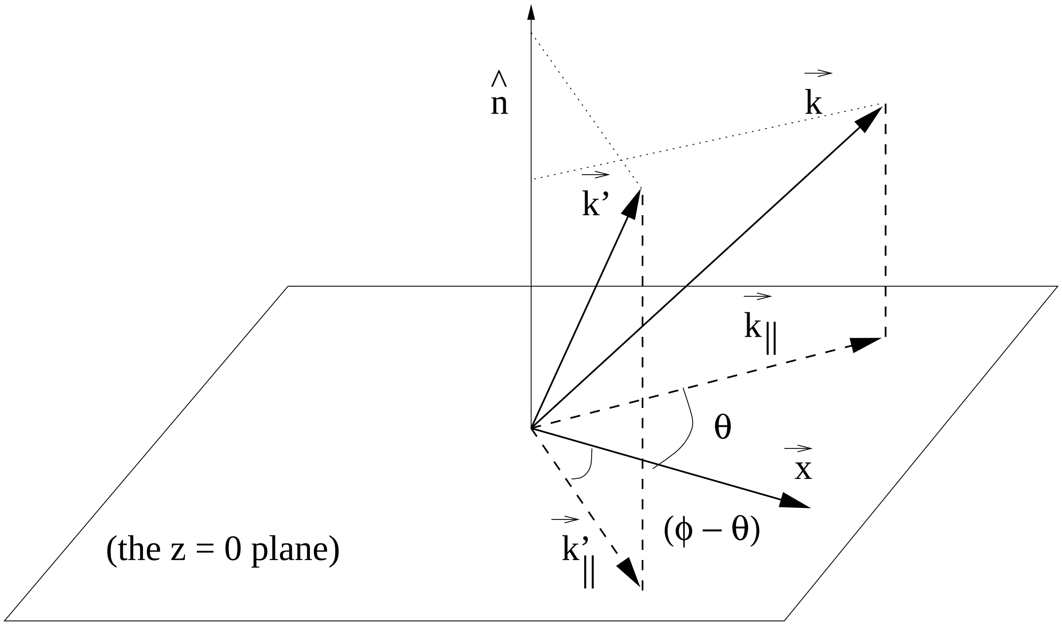
$$\omega = \omega' = \omega''$$

This implies $k = k'' = \sqrt{\epsilon\mu}\,\omega/c$ and $k' = \sqrt{\epsilon'\mu'}\,\omega/c$. Similarly, for the boundary condition to be valid for all values of $\vec{x}$ on the boundary, we need

$$\vec{k}\cdot\vec{x}\Big|_{boundary} = \vec{k}'\cdot\vec{x}\Big|_{boundary} = \vec{k}''\cdot\vec{x}\Big|_{boundary}$$

This implies that all the wave vectors must lie in the same plane which also contains the normal to the surface. This can be understood as follows: Choose a rectangular coordinate system such that the boundary surface is the plane at $z = 0$. Then $\vec{x}\Big|_{boundary} = \vec{x}\Big|_{z=0}$ is an arbitrary vector in the $x - y$ plane. From the origin draw two vectors $\vec{k}$ and $\vec{k}'$ (the construction can be repeated for $\vec{k}''$). Each will have components normal and parallel to the $z = 0$ plane;

$$\vec{k} = \vec{k}_{||} + \vec{k}_{\perp}\,, \qquad \vec{k}' = \vec{k}'_{||} + \vec{k}'_{\perp}$$



By construction $\vec{k}_{\perp} \cdot \vec{x}\Big|_{z=0} = 0$, so we have $\vec{k}_{||} \cdot \vec{x}\Big|_{z=0} = \vec{k}'_{||} \cdot \vec{x}\Big|_{z=0}$ This is simpler because all the vectors involved are now in the same plane. Let $\vec{k}_{||}$ make (fixed) angle ϕ with $\vec{k}'_{||}$ and angle θ with $\vec{x}\Big|_{z=0}$ Then,

$$k_{||}\cos\theta = k'_{||}\cos(\phi - \theta)$$

We want this to be true for all θ with other quantities kept fixed. This is possible only for $\phi = 0$ ($\phi = \pi$ is ruled out since $k_{||}$ and $k'_{||}$ are positive). Thus the plane containing $\vec{k}$ and $\vec{k}'$ is perpendicular to the $z = 0$ plane. The argument can be repeated for $\vec{k}''$.

Now that we know that $\vec{k}$, $\vec{k}'$ and $\vec{k}''$ lie in the same plane together with the normal to the surface, let i, r' and r'' denote the angles that they respectively make with the normal. Choosing a vector $\vec{x}\Big|_{z=0}$ also in the same plane one concludes that since $k = k''$, one has $\cos(\frac{\pi}{2} - i) = \cos(\frac{\pi}{2} - r'')$ and hence, $i = r''$ which is the law of reflection. On the other hand, $k\cos(\frac{\pi}{2} - i) = k'\cos(\frac{\pi}{2} - r')$ or

$$\frac{\sin i}{\sin r'} = \frac{k'}{k} = \sqrt{\frac{\epsilon'\mu'}{\epsilon\mu}}$$

The incident plane wave has velocity $v = \omega/k = c/\sqrt{\epsilon\mu}$ corresponding to the refraction index $n = c/v = \sqrt{\epsilon\mu}$. Therefore, one recovers Snell's law of refraction,

$$\frac{\sin i}{\sin r'} = \frac{n'}{n}$$

The remaining part of the boundary conditions relate the amplitudes of the electric and magnetic fields across the boundary and contain information about the relative intensity of the reflected and refracted waves and their polarizations. These are the dynamical aspects of the problem and will not be discussed further.