Behaviour of $\vec{E}$ and $\vec{B}$ under

Rotations, Parity and Time-reversal

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(Lecture notes – Fawad Hassan, updated Sept 26 2008)

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1 Motivation

It turns out that all physical quantities can be characterized by their behaviour under certain space-time transformations. This greatly restricts the form of physical laws and clarifies the structure of the theory. The space-time transformations that are of interest are those that transform the space-time coordinates $(\vec{x}, t)$, or the coordinate systems in which physical quantities are measured, but which are known not to affect the basic laws of physics. In other words the form of the equations that express the laws of physics should not change under these transformations. In other words, the equations should be *covariant* under the transformations so that the laws of physics are *invariant*. To insure this, the various physical quantities that appear in the equations should transform in very restricted ways.

Here we will consider three such transformations: Rotations, Parity (space-inversion) and Time-reversal which are symmetry transformations in classical mechanics and classical electrodynamics. Our intention is to prepare the ground for our discussion of Lorentz transformations, while in the process shedding more light on the nature of electric and magnetic fields.

1.1 Notation

A vector $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ can also be represented as a column or row matrix:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad x^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}.$$

In this notation the length of a vector is expressed as

$$\vec{x} \cdot \vec{x} = \sum_{i=1}^{3} x_i x_i = x^T x$$

where ^{T} stands for transposition. In general for the scalar product of two vectors one has,

$$\vec{x} \cdot \vec{y} = x^T y = y^T x$$

2 Orthogonal Transformations

Consider a linear transformation $x_i \to \widetilde{x}_i(x)$,

$$\begin{aligned}
\widetilde{x}_1 &= O_{11}\,x_1 + O_{12}\,x_2 + O_{13}\,x_3 \\
\widetilde{x}_2 &= O_{21}\,x_1 + O_{22}\,x_2 + O_{23}\,x_3 \\
\widetilde{x}_3 &= O_{31}\,x_1 + O_{32}\,x_2 + O_{33}\,x_3
\end{aligned}$$

or, in compact form,

$$\widetilde{x}_i = O_{ij}\,x_j \qquad \text{or} \qquad \widetilde{x} = O\,x$$

where O stands for the matrix,

$$O = \begin{pmatrix} O_{11} & O_{12} & O_{13} \\ O_{21} & O_{22} & O_{23} \\ O_{31} & O_{32} & O_{33} \end{pmatrix}$$

The lengths of vectors x and $\widetilde{x}$ are related by

$$\widetilde{x}^T \widetilde{x} = x^T\,O^T O\,x$$

If we demand that O keeps the length of x unchanged, then

$$O^T\,O = \mathbf{1}$$

i.e., O is an orthogonal matrix and $O^{-1} = O^T$. The set of all such transformation matrices forms the group $O(3)$, the group of orthogonal transformations in 3 dimensions. From $O^T\,O = \mathbf{1}$ it follows that $(\det O)^2 = 1$ and therefore,

$$\det O = \pm 1$$

3 Rotations

Elements of $O(3)$ with $\det O = 1$ form the group $SO(3)$, the special orthogonal group in 3 dimensions. These correspond to the usual spatial rotations in 3 dimensional space. Let us denote the elements of this group by matrices R, then $R^T R = 1$ and $\det R = 1$. The simplest example is a rotation, say, in the $x_1 - x_2$ plane, by an angle θ, keeping x_3 unchanged. Then,

$$R_{12} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\widetilde{x} = R_{12} x$, or in components,

$$\begin{aligned}
\widetilde{x}_1 &= \cos\theta x_1 + \sin\theta x_2 \\
\widetilde{x}_2 &= -\sin\theta x_1 + \cos\theta x_2 \\
\widetilde{x}_3 &= x_3
\end{aligned}$$

$SO(3)$ elements are also called *proper* rotations as opposed to *improper* rotations for which $\det O = -1$.

In non-relativistic physics, physical quantities are classified based on their transformation properties under $SO(3)$. They appear as scalars, vectors and higher rank tensors.

3.1 $SO(3)$ scalars

We have seen the length of a vector $\vec{x} \cdot \vec{x}$ is invariant $SO(3)$ rotations. This is an example of a scalar quantity. In general, a quantity $\phi(x)$ is a scalar if under $x \to \widetilde{x} = Rx$ it transforms to a $\widetilde{\phi}(\widetilde{x})$ such that

$$\widetilde{\phi}(\widetilde{x}) \equiv \widetilde{\phi}(Rx) = \phi(x)$$

Examples of $SO(3)$ scalars are the electric potential $\Phi(x)$ and the electric charge density $\rho(x)$ (note that, as we will see later, these are only $SO(3)$ scalars but not Lorentz scalars).

3.2 $SO(3)$ vectors

Under rotations, the position x transforms as $x_i \to \widetilde{x}_i = R_{ij} x_j$. This is the simplest example of a vector. Any quantity transforming in this way is called a vector. In general a general vector $V(x)$ transforms as

$$V_i(x) \to \widetilde{V}_i(\widetilde{x}) = R_{ij} V_j(x)$$

In matrix notation

$$V(x) \to \widetilde{V}(\widetilde{x}) = RV(x) , \quad \text{where} \quad \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Since R_{ij} is a constant matrix, dx/dt transforms in the same way as x. Hence, other vectors are the velocity v, acceleration a, and in particular the force $F = ma$. Since the electric and magnetic fields can be operationally defined in terms of force,

$$\vec{F}_e = q\vec{E} , \qquad \vec{F}_m = \frac{q}{c} \vec{v} \times \vec{B}$$

it follows that they are also vectors (note that in the case of B we have relied on the notion that a cross product relates two vectors to a third vector, hence, since v and F_m are vectors, so is B). Similarly from $\vec{B} = \nabla \times \vec{A}$ it follows that A is also a vector.

3.3 $SO(3)$ Tensors

A rank 2 tensor is an object that transforms as two copies of a vector,

$$T_{ij}(x) \to \widetilde{T}_{ij}(\widetilde{x}) = R_{ik}R_{jl}T_{kl}(x)$$

Note that $R_{ik}R_{jl}T_{kl} = R_{ik}T_{kl}R_{jl} = R_{ik}T_{kl}(R^T)_{lj}$. Hence, using matrix notations for T_{ij} and R_{ij}, one has,

$$\widetilde{T}(\widetilde{x}) = RT(x)R^T$$

Similarly, a rank n tensor transforms like n copies of a vector,

$$\widetilde{T}_{i_1\cdots i_n}(\widetilde{x}) = R_{i_1j_1}\cdots R_{i_nj_n}T_{j_1\cdots j_n}(x)$$

There is no matrix notation for the transformation for $n > 2$. In the language of tensors, scalars are rank 0 tensors and vectors are rank 1 tensors.

4 Parity (Space inversion)

Consider the matrix,

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This is an element of $O(3)$ with $\det P = -1$. It acts on the coordinates as

$$x \to \widetilde{x} = P\,x = -x$$

or in terms of components, $x^i \to -x^i$; hence the name *space inversion*. It is also called a *parity transformation* as it interchanges left- and right-handed coordinate systems . It turns out that all $O(3)$ matrices with $\det O = -1$ can be written as a product of P with $SO(3)$ matrices. While $\vec{E}$ and $\vec{B}$ transform in the same way under $SO(3)$, they behave differently under P. Note that since $\vec{x}$ changes sign under parity, so do the velocity $\vec{v}$, the momentum $\vec{p}$ and hence the force $\vec{F} = md\vec{p}/dt$. Then from $\vec{F}_e = q\vec{E}$ and $\vec{F}_m = (q/c)\vec{v}\times\vec{B}$ it is obvious that under parity,

$$\begin{array}{rcrl} \vec{E} & \to & -\vec{E} & (\text{vector}) \\ \vec{B} & \to & \vec{B} & (\text{pseudo}-\text{vector}) \end{array}$$

Hence $\vec{B}$ does not behave like an *ordinary vector* under parity (the prototype of an ordinary vector is $\vec{x}$ or $\vec{v}$).

To understand the nature of $\vec{B}$ better, let us look at another common pseudo-vector, the angular momentum,

$$\vec{L} = \vec{x}\times\vec{p}$$

Under parity $\vec{x}$ and $\vec{p}$ are odd and therefore $\vec{L} \to \vec{L}$. Note that in terms of components,

$$L_i = \epsilon_{ijk} x_j p_k = \frac{1}{2}\epsilon_{ijk}(x_j p_k - x_k p_j)$$

Defining, $\hat{L}_{jk} = x_j p_k - x_k p_j$, one has[1]

$$L_i = \frac{1}{2}\epsilon_{ijk}\hat{L}_{jk}\,, \quad \text{or} \quad \hat{L}_{ij} = \epsilon_{ijk} L_k$$

This shows that angular momentum can be described in terms of either a vector with components L_i or an antisymmetric rank 2 tensor $\hat{L}_{jk}$. The physical interpretation of this statement becomes obvious from

$$L_1 = \hat{L}_{23}\,, \qquad L_2 = \hat{L}_{31}\,, \qquad L_3 = \hat{L}_{12}$$

Thus an angular momentum along x_1 corresponds to a rotation in the $x_2 - x_3$ plane. Hence, whereas the vector notation for the angular momentum describes it with reference to the *axis* of rotation, the antisymmetric tensor notation refers to the *plane* of rotation [2]. The $\hat{L}_{ij}$ notation for the angular momentum explains its behaviour under P: It essentially transforms as a rank 2 tensor and not as a vector.

One can now look for a similar structure in $\vec{B}$. In fact, since $\vec{B} = \vec{\nabla} \times \vec{A}$, one has

$$B_i = \epsilon_{ijk}\,\partial_j A_k = \frac{1}{2}\epsilon_{ijk}\,(\partial_j A_k - \partial_k A_j)$$

Defining

$$F_{jk} = \partial_j A_k - \partial_k A_j$$

one can write

$$B_i = \frac{1}{2}\epsilon_{ijk}\,F_{jk}\,, \quad \text{or} \qquad F_{jk} = \epsilon_{ijk} B_k$$

More explicitly,

$$B_1 = F_{23}\,, \qquad B_2 = F_{31}\,, \qquad B_3 = F_{12}$$

Thus the magnetic field can be alternatively described in terms of an antisymmetric rank 2 tensor which makes its behaviour under parity manifest,

$$P: \qquad F_{ij} \to F_{ij}$$

(since both A_i and ∂_i are odd under parity). It is this tensor that appears in the Lorentz covariant form of Maxwell's equations.

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[1]

$$\sum_i \epsilon_{imn} L_i = \sum_{ijk} \frac{1}{2}\epsilon_{imn}\epsilon_{ijk}\hat{L}_{jk} = \sum_{ijk}\frac{1}{2}(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})\hat{L}_{jk} = \frac{1}{2}(\hat{L}_{mn} - \hat{L}_{nm}) = \hat{L}_{mn}$$

[2]Note that the description in terms of axis of rotation is not fundamental since it makes sense only in 3 dimensions. In higher than 3 dimensions, one still has planes of rotation but a plane has more than one perpendicular direction and hence there is no unique axis of rotation.

5 Time reversal

This is a transformation that send the time coordinate t to $-t$. The basic laws of classical mechanics and electrodynamics are invariant under this transformation. Under time reversal, $x \to x$ (even) and $t \to -t$ (odd) so that $v \to -v$ and $F \to F$. Then the Lorentz force law implies that $\vec{E}$ is even while $\vec{B}$ is odd. Consequently, ρ and Φ are even but $\vec{J}$ and $\vec{A}$ are odd.