Time-Dependent Fields, Maxwell Equations and Energy Considerations

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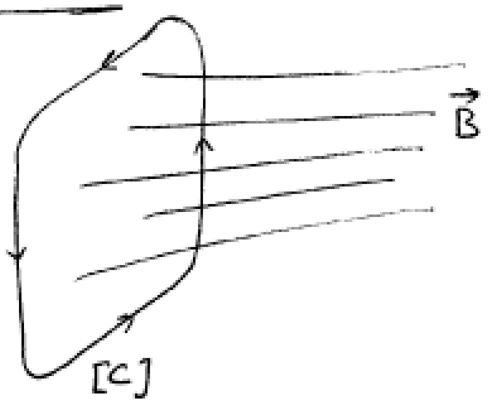
(Scanned class notes – S. F. Hassan)

Time-Dependent Fields and Maxwell Eqns.

We now set out to generalize the electrostatic and magnetostatic equations to time-dependent fields. This generalization is partly contained in Faraday's law of induction. The requirement of consistency with the remaining equations then leads to the full set of time-dependent equations, also called the Maxwell equations.

Faraday's Law of Induction



Faraday's law is an experimental law based on the observation that when a circuit [C] is placed in a varying magnetic field $\vec{B}$, a current starts flowing through it as long as the variation continues. The source of the magnetic field could be either a permanent magnet, or another circuit through which a current flows. The variation of the magnetic field passing through the circuit [C] could be caused either by a variation of $\vec{B}$ (due to the motion of the source or a varying current through the source circuit) or by the motion of the circuit [C] in the field $\vec{B}$.

To quantify this observation, one has to introduce appropriate quantities to describe the current flow in [C] as well as the variation of the magnetic field through the circuit which causes the current flow. A current will flow between two points A and B in a circuit if there is a potential difference between them, i.e., $\int_A^B \vec{E}'\cdot\vec{dl} \neq 0$. Here $\vec{E}'$ is the electric field acting on the charges in [C] (the reason for using a "prime" will be explained later). For charges to flow around a closed circuit [C], there has to exist a driving force around the circuit which does work in taking the charges around. If $\vec{E}'$ is the electric field seen by the charges in [C], then this work is

$$\xi = \oint_{[C]} \vec{E}'\cdot\vec{dl} \neq 0$$

ξ is called the electromotive force. Note that the flow of a current through a closed circuit already implies that $\vec{\nabla}\times\vec{E}' \neq 0$. The electromotive force ξ quantifies the agent that causes the current flow in the circuit.

The circuit [C] treads a magnetic flux

$$\mathfrak{F} = \int_S \vec{B}\cdot\vec{ds}$$

Faraday's experiments showed that the induced current in [C] depended on the variation of the flux through S which is the area of the surface bounded by [C]. The relation is,

$$\boxed{\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\mathcal{F}}$$

The -ve sign is required by Lenz's law which states that the current in [C] flows in such a way (direction) that the magnetic field produced by it opposes the change in $\mathcal{F}$ through S. In terms of $\vec{E}'$ and $\vec{B}$, Faraday's law reads,

$$\boxed{\oint_{[C]} \vec{E}' \cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S}}$$

This statement is valid irrespective of whether the EMF is produced by a varying $\vec{B}$ (i.e., $\partial\vec{B}/\partial t \neq 0$) or by a moving circuit [C], or both. In particular, let the circuit move with velocity $\vec{v}$ relative to the observer (in the lab frame), and also $\partial\vec{B}/\partial t \neq 0$. If the lab frame observer measures electric and magnetic fields $\vec{E}$ and $\vec{B}$ at the location of the circuit [C], then the electrons in [C] will experiance a force per unit charge given by

$$\vec{E}' = \vec{E} + \frac{1}{c}\vec{v} \times \vec{B}$$

due to their motion with [C]. Clearly, $\vec{E}'$ is the field that pushes the electrons around the circuit causing the current. Hence it is $\vec{E}'$ that appears in Faraday's law. On the other hand, it is more natural to find relations between quantities measured in the same frame, like $\vec{E}$ and $\vec{B}$ and not $\vec{E}'$ and $\vec{B}$ which are measured in different frames. Since the motion of the circuit can always be described in terms of the Lorentz force law, we can concentrate on the case of a stationary circuit [C] with a time dependent magnetic field. Then,

$$\vec{E}'\Big|_{\vec{v}=0} = \vec{E} \qquad \text{and} \qquad \frac{d}{dt}\Big|_{\vec{v}=0} = \frac{\partial}{\partial t}$$

So Faraday's law becomes,

$$\oint_{[C]} \vec{E}\cdot d\vec{l} = -\frac{1}{c}\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

or

$$\int_S d\vec{s}\cdot\left(\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right)$$

Since this is true for any circuit shape, we have,

$$\boxed{\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0}$$

This is the time dependent version of the electrostatic equation $\vec{\nabla}\times\vec{E} = 0$.

Maxwell's Displacement Current and Maxwell Equations

Thus far, we have obtained the following eqns. governing the behaviour of electromagnetic fields :

① $\nabla \cdot \vec{D} = 4\pi\rho$

② $\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} \qquad (\Rightarrow \nabla \cdot \vec{J} = 0)$

③ $\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$

④ $\nabla \cdot \vec{B} = 0$

Maxwell noticed that eqn. ② (Ampere's law) was valid only for steady state phenomena as it leads to $\nabla \cdot \vec{J} = 0$. He obtained the modification needed in the case of time-dependent fields by requiring consistency with the continuity equation. Using eqn ①, the continuity eqn. can be rewritten as

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\vec{J} + \frac{1}{4\pi}\frac{\partial \vec{D}}{\partial t} \right) = 0$$

This gives us a divergenceless quantity which generalizes $\nabla \cdot J = 0$ to time-dependent cases when $\partial\rho/\partial t$ is no longer zero. Maxwell replaced $\vec{J}$ in ② by $\vec{J} + \frac{1}{4\pi}\partial\vec{D}/\partial t$ making it consistent with the continuity equation. The new term $\frac{1}{4\pi}\frac{\partial \vec{D}}{\partial t}$ is called Maxwell's displacement current.

We can now write the full set of Maxwell eqns as

$$\boxed{\begin{aligned} &1. \quad \nabla\cdot\vec{D} = 4\pi\rho \\ &2. \quad \nabla\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J} \\ &3. \quad \nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \\ &4. \quad \vec{\nabla}\cdot\vec{B} = 0 \end{aligned}}$$

Some features of these equations will be discussed below. Note that 3 and 4 are not affected by the presence of the medium. One way to understand this is to remember that the modifications in 1 and 2 came about as a result of taking into account the polarization charge density $\nabla\cdot\vec{P}$ and the magnetization current $\nabla\times\vec{M}$ by adding them to ρ and $\vec{J}$. In other words, the electric and magnetic polarizations of the medium modified the source terms ρ and $\vec{J}$ in 1 and 2. This modification was then absorbed in the fields $\vec{E}$ and $\vec{B}$ leading to $\vec{D}$ and $\vec{H}$. However, equations 3 and 4 do not have source terms to absorb polarization effects, and hence remain unmodified.

Scalar and Vector Potentials and Gauge Invariance :

The condition of the absence of monopoles has not changed allowing us to again express $\vec{B}$ in terms of a vector potential $\vec{A}$:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Longleftrightarrow \quad \boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

On the other hand, $\vec{E}$ can no longer be written as $-\nabla \Phi$, since $\nabla \times (\nabla \Phi) = 0$, while $\nabla \times \vec{E} \neq 0$. However,

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$$

becomes,

$$\vec{\nabla} \times \left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0$$

Hence, we can write $\left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = -\nabla \Phi$.
Therefore,

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad \Longleftrightarrow \quad \boxed{\vec{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}}$$

The expressions for $\vec{E}$ and $\vec{B}$ in terms of Φ and $\vec{A}$ are the solutions to eqns. 3) and 4), which after the introduction of potentials are trivially satisfied and become redundant.

Gauge Invariance:

The expressions

$$\vec{B} = \nabla \times \vec{A} \quad , \quad \vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

do not associate a unique $\vec{A}$ and Φ to $\vec{E}$ and $\vec{B}$. In fact, changing $\vec{A}$ and Φ as

$$\left.\begin{aligned} \vec{A} &\longrightarrow \vec{A}' = \vec{A} + \nabla\Lambda \\ \Phi &\longrightarrow \Phi' = \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \end{aligned}\right\} \text{Gauge transformation}$$

will keep $\vec{E}$ and $\vec{B}$ unchanged. This is referred to as gauge invariance. This comes very handy when we want to solve problems in terms of $(\vec{A}, \Phi)$. Often, by making an appropriate gauge transformation, the problem can be considerably simplified without changing its physical content.

Maxwell's Equations in terms of Potentials

As we have seen above, the introduction of potentials already solves the two sourceless Maxwell's equations. The two equations with sources can now be written in terms of the potentials by substituting for $\vec{E}$ and $\vec{B}$ in terms of the potentials $\vec{A}$ and Φ. This gives (for free space),

$$\boxed{\begin{aligned} &\nabla^2\Phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho \\ &\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = -\frac{4\pi}{c}\vec{J} \end{aligned}} \quad \text{(free space)}$$

To write the second eqn. we have used

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) .$$

As such, these eqn's look more complicated than the original ones in terms of $\vec{E}$ and $\vec{B}$. However, they can be simplified by making appropriate gauge transformations. One choice is to use the freedom in shifting $\vec{A}$ and Φ ($\vec{A} \to \vec{A} + \nabla\Lambda$, $\Phi \to \Phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$) and choose them such that

$$\boxed{\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0} \quad \text{(Lorentz gauge)}$$

This is called the Lorentz gauge. With this, the above equations decouple:

$$\boxed{\begin{aligned} &\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho \\ &\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J} \end{aligned}}$$

(Maxwell's eqn's in terms of potentials in free space - Lorentz gauge.)

Once the equations are written in this form, we are not allowed to perform general gauge transformations any more, since they would change the form of the equations. The only gauge transformations that keep the form of these equations unchanged are the ones that keep the Lorentz gauge condition unchanged, i.e.,

$$\nabla\cdot\vec{A}' + \frac{1}{c}\frac{\partial\varphi'}{\partial t} = 0 \quad \& \quad \nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0.$$

for $\vec{A}' = \vec{A} + \nabla\Lambda$ and $\Phi' = \Phi - \frac{1}{c}\partial\Lambda/\partial t$, this amounts to

$$\boxed{\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0}$$

These give the residual gauge transformations after fixing the Lorentz gauge.

Energy Injected into a Circuit by Electric and Magnetic Fields:

Consider a particle of charge q moving with a velocity $\vec{v}$ in an electromagnetic field described by $\vec{E}$ and $\vec{B}$. It will have an instantaneous kinetic energy $K_q = \frac{1}{2} m \vec{v}\cdot\vec{v}$ and experiance a force $\vec{F} = q(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})$. The resulting acceleration will change K_q as

$$\frac{\partial K_q}{\partial t} = m\vec{v}\cdot\vec{a} = q\left(\vec{v}\cdot\vec{E} + \frac{1}{c}\vec{v}\cdot(\vec{v}\times\vec{B})\right)$$

By definition $\vec{v}\times\vec{B}$ is perpendicular to the plane defined by $\vec{v}$ and $\vec{B}$, and in particular to the vector $\vec{v}$. Hence, $\vec{v}\cdot(\vec{v}\times\vec{B}) = 0$, and one has

$$\boxed{\frac{\partial K_q}{\partial t} = q\,\vec{v}\cdot\vec{E}}$$

The magnetic field does not do work on moving charges (magnetic force is normal displacement). Note however that a time dependent magnetic field will give rise to an electric field which will in turn accelerate a charged particle.

Let us now consider a current distribution $\vec{J}(x) = \vec{v}\rho(\vec{x})$, where $\rho(x)$ is the charge density at $\vec{x}$ moving with velocity $\vec{v}$. Note that

$\rho(x)$ here stands for the density of moving charges (e.g. electrons in conduction band). The medium may also contain stationary charges of density $-\rho(x)$; giving rise to a vanishing net charge density. The current $\vec{J}(x)$ will however not vanish, depending only on the moving charges. The (moving) charge contained in a small volume ΔV around the point $\vec{x}$ is $\rho(x)\,\Delta V$ and the power injected into ΔV by an electric field $\vec{E}$ is

$$\frac{\partial(\Delta K)}{\partial t} = \rho(x)\,\vec{v}\cdot\vec{E}\,\Delta V = \vec{J}\cdot\vec{E}\,\Delta V$$

The power injected into the whole current distribution is then $\int_V \partial(\Delta K)/\partial t \, dV$, or

$$\boxed{\frac{dK}{dt} = \int_V \vec{J}\cdot\vec{E}\, d^3x}$$

What happens to this power in the circuit? There are two main possibilities depending on whether steady state has been achieved or not.

1) As we have seen the power transfer results in a change in the kinetic energy, and hence velocity of charge carriers. Thus instantaneously the power injected changes the current through the circuit. If we want to keep the current unchanged, then we have to perform work to counter

the effect of $\vec{E}$. As we will see below, this work canbe stored as magnetic energy in the system.

2) In steady state, by definition the current remains constant, in spite of the continuous power injection. The reason is that the extra energy is lost in collisions between the electrons and fixed charges. This lost energy finally appears as heat. To verify this, let us consider the current through a thin wire. Then $d^3x = d\vec{l}\cdot d\vec{s}$. We assume that $\vec{E}$ remains essentially constant over the cross-section of the wire. Also, for a thin wire, $\vec{J}$ is parallel to $d\vec{l}$. Hence, for the power injected we get the expression:

$$W = \frac{\partial K}{\partial t} = \int_V \vec{J}\cdot\vec{E}\, d^3x = \int_S \oint_l (\vec{J}\cdot\vec{E})(d\vec{l}\cdot d\vec{s})$$

$$= \oint_l (d\vec{l}\cdot\vec{E}) \int_S \vec{J}\cdot d\vec{s} = I \oint_l d\vec{l}\cdot\vec{E}$$

where I is the total current through the wire. Remember that $\int_A^B d\vec{l}\cdot\vec{E}$ gives the potential difference between points A and B. Therefore, $\oint_l d\vec{l}\cdot\vec{E}$ is the potential difference around the circuit or the applied voltage V. Hence,

$$W = IV = I^2R \qquad (\text{for } V = IR\text{, Ohm's law})$$

We recognize this as the ohmic heat loss.

Energy Density in a Magnetic Field

We have already seen that an electrostatic field $\vec{E}$ has energy density $\frac{1}{8\pi}\vec{E}\cdot\vec{E}$ in free space. In a dielectric medium the same derivation leads to energy density $\frac{1}{8\pi}\vec{E}\cdot\vec{D}$. This result was derived by associating the field energy with the self-interaction energy of a charge distribution $\rho(x)$. It was computed as the work done to assemble the charge distribution. Equivalently it was the work done to build up $\rho(x)$ by adding increments $\delta\rho(x)$ slowly and step by step, at various space points $\vec{x}$; or, $\rho(x) = \int_0^\rho \delta\rho(x)$. All this could be done within the framework of electrostatics.

We would like to use a similar procedure to compute the energy density in a magnetic field $\vec{B}$. The difference is that even for magnetostatic fields, the computation involves time-dependent effects. That is the reason we have not taken this up untill now. Before describing the computation, let us first see how these time-dependent effects arise:

A magnetic field $\vec{B}$ is always produced by a current distribution $\vec{J}(\vec{x})$. Hence its energy density can be computed by calculating the work needed to set up the current distribution $\vec{J}(x)$.

We consider building $\vec{J}(x)$ up over the volume V by adding small increments $\delta\vec{J}(x)$ at each space point $\vec{x}$, step by step, such that at each step $\vec{J}$ at all other space points is kept fixed. This is repeated untill the final configuration is reached. To achieve this, we have to perform work on the system for the following reason: Consider adding an increment $\delta\vec{J}(x_0)$ at some point $\vec{x}_0$. This change in the current density at $\vec{x}_0$ changes the associated magnetic field which in turn induces an electric field by Faraday's law. The induced electric field will tend to change the current flow at other points $\vec{x} \neq \vec{x}_0$ by injecting a power $\vec{J}\cdot\vec{E}$ into the current distribution. To keep $\vec{J}(\vec{x} \neq \vec{x}_0)$ unchanged, we have to supply extra energy to the system to balance the effect of $\vec{J}\cdot\vec{E}$. In this way, at the end of the step we end up with increased current at $\vec{x}_0$, $\vec{J}(\vec{x}_0) + \delta\vec{J}(\vec{x}_0)$, keeping the rest of the system unchanged. The larger current flow is now associated with a larger magnetic field by Ampere's law. The work done to increase the current flow in this way is stored in the magnetic field as potential energy. It is now clear that even though we reach a steady state configuration at the end, the energy density of the magnetic field originates in time -

-dependent effects (work done to balance the effect of Faraday's induction).

Note that besides the above work, we also have to continuously supply energy to the system simply to sustain the current flow and counter resistive losses in the medium. This part of the work done on the system will however appear as Ohmic heat and therefore should be neglected when calculating the energy stored in the magnetic field. Finally we will include all these contributions when discussing the Poynting theorem.

Let us now explicitly compute the energy density in the magnetic field. For this purpose, let us consider two current distributions $\vec{J}_1(\vec{x}_1)$ and $\vec{J}_2(x_2)$ which are non-zero over volumes V_1 and V_2. The coordinates $\vec{x}_1$ and $\vec{x}_2$ span these volumes, respectively.

Let us denote the magnetic fields at some point $\vec{x}$, associated with these current densities as $\vec{B}_1(\vec{x})$ and $\vec{B}_2(\vec{x})$. We assume that the system has no net charge density ($\rho_{net} = 0$) and as a result, in

$\vec{J}_1(\vec{x}_1)$ $\qquad$ $\vec{J}_2(\vec{x}_2)$

steady state, $\vec{J}_1$ and $\vec{J}_2$ do not give rise to any electric fields, $\vec{E}_1 = 0$, $\vec{E}_2 = 0$. The expressions for the magnetic fields can be easily written down using the vector potentials,

$$\vec{B}_{1,2}(\vec{x}) = \vec{\nabla} \times \vec{A}_{1,2}(\vec{x}).$$

From our discussion in magnetostatics we know that the vector potential $\vec{A}_2(\vec{x})$ is given in terms of its current source $\vec{J}_2(\vec{x}_2)$ as,

$$\vec{A}_2(\vec{x}) = \frac{\mu}{c} \int_{V_2} \frac{\vec{J}_2(\vec{x}_2)}{|\vec{x} - \vec{x}_2|} \, d^3x_2$$

There is a similar expression for $\vec{A}_1(\vec{x})$. The presence of "μ" indicates that we are allowing for the possibility of magnetic media with linear responce.

Let us now vary the current $\vec{J}_2(\vec{x}_2)$ by an amount $\delta\vec{J}_2(\vec{x}_2)$, over some time interval δt. This causes a variation in $\vec{A}_2$ and hence in $\vec{B}_2$. The change in $\vec{B}_2$ induces an electric field $\vec{E}_2(x)$ through Faraday's law. The induced electric field can be easily expressed in terms of the vector potential $\vec{A}_2$. In general, we know that $\vec{E} = -\nabla\Phi - 1/c \, \partial\vec{A}/\partial t$. In the present example, there is no net charge to serve as a source for Φ. Hence, $\nabla\Phi = 0$ and the induced electric field is given

by,
$$\vec{E}_2(\vec{x}) = -\frac{1}{c}\frac{\partial \vec{A}_2(\vec{x})}{\partial t}$$

This field is non-zero at the location of the first current distribution, $\vec{E}_2(\vec{x}_1) \neq 0$, and hence injects power into it. As we have seen, the injected power is

$$P = \frac{\partial K}{\partial t} = \int_{V_1} \vec{J}_1(\vec{x}_1) \cdot \vec{E}_2(\vec{x}_1)\, d^3x_1 .$$

Instantaneously, it tends to change the current $\vec{J}_1$. To keep this constant (so that the net increase in current density is $\delta\vec{J}_2$) we have to supply energy at rate $-P$ to the system to counter the effect of variation $\delta\vec{J}_2$. Hence the system gains energy at a rate

$$\frac{\partial W_{12}}{\partial t} = -\int_{V_1} \vec{J}_1(\vec{x}_1) \cdot \vec{E}_2(\vec{x}_1)\, d^3x_1$$
$$= \frac{1}{c}\int_{V_1} \vec{J}_1(\vec{x}_1) \cdot \frac{\partial \vec{A}_2(\vec{x}_1)}{\partial t}\, d^3x_1$$

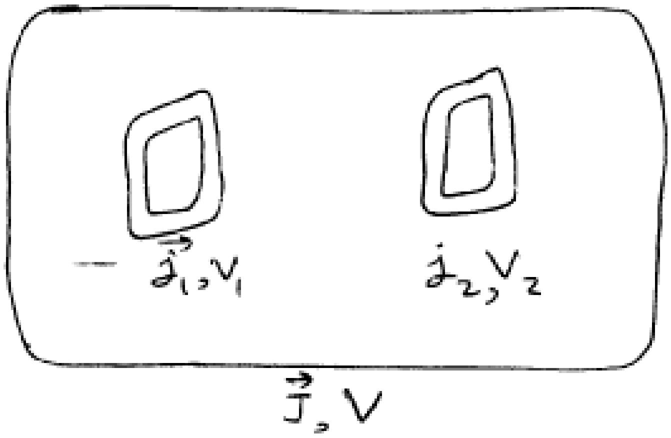
The total energy increase as a result of variation $\delta\vec{J}_2$ (over some time interval δt) is

$$\boxed{\delta W_{12} = \frac{1}{c}\int_{V_1} \vec{J}_1(\vec{x}_1) \cdot \delta\vec{A}_2(\vec{x}_1)\, d^3x_1}$$

or,
$$\delta W_{12} = \frac{\mu}{c^2}\iint_{V_1 V_2} \frac{\vec{J}_1(\vec{x}_1) \cdot \delta\vec{J}_2(\vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|}\, d^3x_1\, d^3x_2$$

We now consider a current distribution $\vec{J}(x)$ over some volume V and regard $\vec{j}_1(x_1)$ and $\vec{j}_2(x_2)$ as two small current elements within $\vec{J}(x)$. This means that when $\vec{x}$ is in the small volume V_1, i.e., $\vec{x} = \vec{x}_1$, then $\vec{J}(x_1) = \vec{j}_1(x_1)$, and when $\vec{x}$ is in V_2, then $\vec{J}(x_2) = \vec{j}_2(x_2)$.

Let us again vary $\vec{j}_2(x_2)$ by an amount $\delta\vec{j}_2$ within V_2 and keep the current $\vec{J}$ in the rest of V unchanged. The total increase in the energy of the system is then



$$\delta W = \int_V \delta W_{12}$$

This integration can be implemented by replacing the x_1-integration over V_1 in the expression for δW_{12} by the x-integration over V. Hence,

$$\boxed{\delta W = \frac{1}{c} \int_V \vec{J}(\vec{x}) \cdot \delta \vec{A}(\vec{x})\, d^3x .}$$

δW can be written entirely in terms of the fields by using $\nabla \times \vec{H} - \frac{1}{c}\, \partial \vec{D}/\partial t = 4\pi/c\, \vec{J}$ to substitute for $\vec{J}(x)$. However, note that in δW the effect of time variation has been taken into account in $\delta \vec{A}$. The $\vec{J}(\vec{x})$ factor (in δW) replaces $\vec{J}_1(x_1)$ which was kept in steady state by supplying extra energy to the system. Hence, it reasonable to regard $\vec{J}(\vec{x})$ as a steady state current ($\nabla \cdot \vec{J} = 0$) and use

$$\vec{J}(x) = \frac{c}{4\pi} \nabla \times \vec{H}$$

Substituting this in δW, we get

$$\delta W = \frac{1}{4\pi} \int (\nabla \times \vec{H}) \cdot \delta \vec{A} \, d^3x .$$

The curl operator $(\nabla \times)$ can be moved over to $\delta \vec{A}$ using the identity

$$\nabla \cdot (\vec{P} \times \vec{Q}) = \vec{Q} \cdot (\nabla \times \vec{P}) - \vec{P} \cdot \nabla \times \vec{Q}$$

along with the divergence theorem and ignoring the surface term. This gives,

$$\delta W = \frac{1}{4\pi} \int \vec{H} \cdot (\nabla \times \delta \vec{A}) \, d^3x$$

or

$$\boxed{\delta W = \frac{1}{4\pi} \int \vec{H} \cdot \delta \vec{B} \, d^3x}$$

$$\left(\text{since } \vec{B} = \nabla \times \vec{A} \right).$$

If the medium has a linear magnetic responce, then $\vec{H} = \frac{1}{\mu}\vec{B}$ and

$$\delta W = \frac{1}{4\pi\mu}\int \vec{B}\cdot\delta\vec{B}\ d^3x$$

$$= \frac{1}{8\pi\mu}\int \delta(\vec{B}\cdot\vec{B})\ d^3x$$

The change in the energy δW is accompanied by a change $\delta\vec{B}$ in the magnetic field. If we regard the current distribution $\vec{J}(x)$ (which gives rise to the magnetic field $\vec{B}$) as being built up step by step starting with $\vec{J}=0$ (corresponding to $\vec{B}=0$), then the total energy supplied to the system is given by,

$$W = \int_{B=0}^{B}\delta W = \frac{1}{8\pi\mu}\int_{B=0}^{B}\left(\int \delta(\vec{B}\cdot\vec{B})d^3x\right)$$

$$\boxed{W = \frac{1}{8\pi\mu}\int \vec{B}\cdot\vec{B}\ d^3x}$$

As explained earlier, this of course do not include the energy dissipated into heat. The energy W is stored in the magnetic field which therefore has energy density

$$\boxed{\omega = \frac{1}{8\pi\mu}\vec{B}\cdot\vec{B} = \frac{1}{8\pi}\vec{H}\cdot\vec{B}}$$

The Poynting Theorem and Energy Conservation :

Consider a region of space with current distribution $\vec{J}(x)$ and electric and magnetic fields $\vec{E}$ and $\vec{B}$. For definiteness we consider some volume V outside which $\vec{J}(x)$ is zero, so that no charge particles leave V. We know that the energy density stored in $\vec{E}$ and $\vec{B}$ is given by (at least in the steady state)

$$u(x) = \frac{\epsilon}{8\pi} \vec{E}\cdot\vec{E} + \frac{1}{8\pi\mu} \vec{B}\cdot\vec{B}$$

We also know that the electric field injects energy into the current distribution. This energy transfer per unit volume is at a rate

$$\frac{dw}{dt} = \vec{E}\cdot\vec{J}$$

As a result of this energy loss the energy density of the fields may change in time, $\partial u/\partial t \neq 0$. However, this is not the end of the story. The electric and magnetic fields may also bring in some energy into the volume V which could partly contribute to the field energy density u and partly to the loss $\frac{\partial w}{\partial t}$. The sum total of these three forms of energy should be conserved. So far we do not have an expression for the energy that flows into the volume V due to non-vanishing electric and magnetic fields. This expression is identified by the Poynting theorem. This is obtained by simply eliminating $\vec{J}$ in $\vec{J}.\vec{E}$ in favour of the fields using Maxwell's equations. Note that

$$\vec{J} = \frac{c}{4\pi}\left(\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t}\right)$$

so that,

$$\vec{J}.\vec{E} = \frac{c}{4\pi}\left(\vec{E}.\nabla\times\vec{H}\right) - \frac{1}{4\pi}\frac{\partial \vec{D}}{\partial t}.\vec{E}$$

The identity $\nabla \cdot (\vec{P}\times\vec{Q}) = (\nabla\times\vec{P}).\vec{Q} - \vec{P}.(\nabla\times\vec{Q})$ implies that,

$$\vec{E}.(\nabla\times\vec{H}) = (\nabla\times\vec{E}).\vec{H} - \nabla.(\vec{E}\times\vec{H})$$

$$= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.H - \nabla.(\vec{E}\times\vec{H}).$$

Then,

$$\vec{J}.\vec{E} = -\frac{1}{4\pi}\frac{\partial B}{\partial t}.H - \frac{1}{4\pi}\frac{\partial \vec{D}}{\partial t}.\vec{E} - \frac{c\nabla.}{4\pi}(\vec{E}\times\vec{H})$$

For linear media ($\vec{B} = \mu\vec{H}$, $\vec{D} = \epsilon\vec{E}$),

$$\boxed{-\vec{J}.\vec{E} = \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon\,\vec{E}.\vec{E} + \frac{1}{\mu}\vec{B}.\vec{B}\right) + \frac{c}{4\pi}\nabla.(\vec{E}\times\vec{H})}$$

This is the Poynting theorem and the quantity

$$\vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{H})$$

is called the Poynting vector. It represents the energy flow into the volume V. More precisely, $\nabla \cdot \vec{S}$ is the energy per unit volume flowing into the system by virtue of the electric and magnetic fields. The Poynting theorem has the integral form,

$$-\int_V \vec{J}\cdot\vec{E}\, d^3x = \frac{\partial}{\partial t}\int_V \left(\frac{\epsilon E^2}{8\pi} + \frac{B^2}{8\pi\mu}\right) d^3x + \oint_{\partial V} \vec{S}\cdot da$$

"∂V" denotes the boundary of volume V and $\oint_{\partial V} \vec{S}\cdot d\vec{a}$ represents the total energy flux entering the system. Thus $\vec{S}$ represents the flow of energy per unit area per unit time.

Note that $\vec{J}\cdot\vec{E}$ appears as the mechanical energy of the charge carriers, hence we can use the notation,

$$\frac{dE_{mech}}{dt} = \int_V \vec{J}\cdot\vec{E}\, d^3x$$

Also, $\quad E_{field} = \int_V \left(\frac{\epsilon E^2}{8\pi} + \frac{B^2}{8\pi\mu}\right) d^3x$

Then the Poynting theorem can be written as

$$\boxed{\frac{d}{dt}\left(E_{mech} + E_{field}\right) = -\oint_{\partial V} \vec{S}\cdot d\vec{a}}$$