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(Scanned class notes – S. F. Hassan)

Magnetostatics :

Although electric and magnetic phenomena may look similar superficially, there are basic differences between the two. The major difference stems from the fact that there are no magnetic charges in nature. Magnetism is entirely associated with the motion of electric charges, and hence with currents. As in the case of electric forces, magnetic forces can be described by the notion of a magnetic field. A charged particle in motion produces a magnetic field in its vicinity (of course what matters is relative motion). Another charged particle moving in this vicinity experiances a force due to the field. Therefore, as in the case of electricity, the magnetic field can be defined in terms of the force that a moving charge experiances when moving in it. The force law (known as the Lorentz force) is known experimentally and is given by

$$\boxed{\vec{F} = q\vec{E} + q\,\frac{\vec{V}}{c} \times \vec{B}}$$

The first term is known to us from electrostatics. It tells us how a static charge is affected by an electric field. There, we also know the other half of the story: how the electric field is set up by charge distributions. As for magnetism, the Lorentz force tells us how a moving charged particle is affected by an existing magnetic field. It thus enables us to measure $\vec{B}$ in terms of force experienced by a moving charge. However, as yet we do not know the other half of the story: How a magnetic field is setup by moving charges (or currents). This will be the subject of the following lessons.

Before considering the most general case, we first describe magnetostatics. This deals with magnetic fields associated with current flows in steady state. A steady state is characterized by non-varying charge densities, $\partial \rho / \partial t = 0$. In terms of currents, this implies $\nabla \cdot \vec{J} = 0$. To understand this, we describe the continuity equation below.

The Continuity Equation

The continuity equation in electrodynamics is a statement of charge conservation which, at the classical level, says that charges cannot be created or destroyed, but only moved around. A more precise statement, going beyond classical physics, is that net charge is always conserved, although equal amounts of positive and negative charges could be created or destroyed. Charge conservation, as stated above is an "integral" statement applying to net charges within macroscopic volumes. The continuity eqn. is a differential form of this statement. We will first obtain this from a macroscopic point of view and then from a microscopic approach.

The macroscopic approach: By definition, the electric current I, say through a surface ΔS, is the net amount of charge crossing the surface per unit time (of course, evaluated for a small time interval). Let us consider a volume V bounded by a surface S.

We choose the convention

or

that, for a closed surface S, the current I_S through S is positive if a net positive charge leaves the volume V. Let us denote the total amount of charge in V by $Q_V(t)$ which could change in time. Charge conservation then tells us that a variation of Q_V should be accompanied by a flow of charge across the surface S, i.e.,

$$\boxed{\frac{\partial Q_V}{\partial t} = -I_S} \qquad (S = \partial V)$$

The -ve sign is required by our convention: If net +ve charge leaves the volume V, then $\frac{\partial Q_V}{\partial t} < 0$ while $I_S > 0$.

Let ρ and $\vec{J}$ denote charge and current densities, respectively. Then,

$$\frac{\partial}{\partial t}\int_V \rho\, dv = -\oint_S \vec{J}\cdot d\vec{s} = -\int_V \nabla\cdot\vec{J}\, dv.$$

Since V is time independent and can be chosen arbitrarily, one gets the continuity equation,

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$

This is the differential statement of charge conservation.

In a steady state flow, $\partial\rho/\partial t = 0$ and therefore,

$$\boxed{\vec{\nabla}\cdot\vec{J} = 0} \qquad (\text{steady state}).$$

<u>The Microscopic Approach</u>: This approach, although more involved, is not much more illuminating and therefore could be skipped. Let us first consider the relation between microscopic and macroscopic charge and current densities. The microscopic densities are given by

$$\rho_m(\vec{x},t) = \sum_a q_a\, \delta(\vec{x} - \vec{x}_a'(t))$$

$$\vec{J}_m(\vec{x},t) = \sum_a q_a \frac{d\vec{x}_a'}{dt}\, \delta(\vec{x} - \vec{x}_a'(t))$$

where the sum runs over the no. of charges. Let us consider a small volume ΔV which is still large on molecular scales. Then, the macroscopic charge density ρ can be defined as:

$$q_{\Delta V} = \int_{\Delta V} dv\, \rho(\vec{x},t) \quad ,$$

$$\rho(\vec{x},t) = \lim_{\Delta V \to 0} \left(\frac{q_{\Delta V}}{\Delta V} \right)$$

Also, consider a surface ΔS through which the charges pass in time Δt. The macroscopic current density is given by

$$\vec{J}(\vec{x},t) = \lim_{\substack{\Delta S \to 0 \\ \Delta t \to 0}} \frac{1}{\Delta S} \frac{1}{\Delta t} \int_{\Delta S} \int_{\Delta t} \vec{J}_m \cdot d\vec{S}\, dt$$

Now, consider,

$$\frac{\partial \rho_m}{\partial t} = \sum_a q_a \frac{dx'^i_a}{dt} \frac{\partial}{\partial x'^i_a} \delta(\vec{x}-\vec{x}'_a)$$

$$= -\sum_a q_a \frac{dx'^i_a}{dt} \frac{\partial}{\partial x^i} \delta(\vec{x}-\vec{x}'_a)$$

$$= -\vec{\nabla}\cdot \sum_a q_a \frac{d\vec{x}'_a}{dt} \delta(\vec{x}-\vec{x}'_a) = -\vec{\nabla}\cdot\vec{J}_m$$

Hence,

$$\boxed{\frac{\partial \rho_m}{\partial t} + \vec{\nabla}\cdot\vec{J}_m = 0}$$

This is the continuity equation for the microscopic variables. The macroscopic version can be obtained by averaging over ΔV and Δt. Then,

$$\frac{1}{\Delta t}\int_{\Delta t}\int_{\Delta V} \vec{\nabla}\cdot\vec{J}_m \, dv\, dt = \frac{1}{\Delta t}\int_{\Delta t}\int_{\Delta S} \vec{J}_m\cdot d\vec{s}\, dt = \sum_\mu \int_{\Delta t}\int_{\Delta \vec{S}_\mu} \vec{J}_m\cdot d\vec{S}_\mu\, dt$$

$$= \sum_\mu \Delta S_\mu \left(\frac{1}{\Delta t}\frac{1}{\Delta S_\mu}\int_{\Delta t}\int_{\Delta S_\mu} \vec{J}_m\cdot d\vec{S}_\mu\, dt\right)$$

$$= \sum_\mu \Delta\vec{S}_\mu\cdot\vec{J} = \oint_{\Delta S} \vec{J}\cdot dS = \int_{\Delta V} \vec{\nabla}\cdot\vec{J}\, dv.$$

$\Rightarrow$

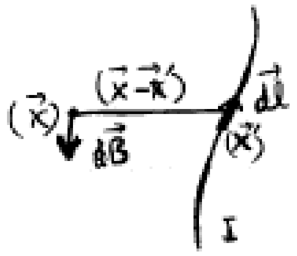
$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0}$$

Biot and Savart Law :

The motion of electric charges in a current carrying conductor sets up a magnetic field around it. Biot and Savart determined the relation between the current and the magnetic field it sets up, experimentally (although the original work did not involve the notion of a field which was introduced later).

consider a thin wire carrying current I. Now, consider a segment $d\vec{l}$ of the wire at the position $\vec{x}'$. The elemental magnetic field $d\vec{B}$ at $\vec{x}$ due to current I flowing through $d\vec{l}$ is given by

$$d\vec{B}(\vec{x}) = \frac{1}{c}\, I\, \frac{d\vec{l} \times (\vec{x} - \vec{x}')}{|x - x'|^3}$$



The total magnetic field is obtained by integrating over the length of the wire:

$$\boxed{\vec{B}(x) = \frac{I}{c} \int_l \frac{d\vec{l} \times (\vec{x} - \vec{x}')}{|x - x'|^3}}$$

This can be evaluated once the geometry of the wire is given. The force that a second current carrying conductor experiances when placed inside the field $\vec{B}$ can be computed from the Lorentz force law. (The evaluation of this for a straight wire is left as a homework assignment)

Let us now consider current carrying conductors which are not infinitesimally thin. If the conductor has cross-sectional area S through which a current of local current density $\vec{J}$ (i.e. current/unit surface area) flows, then

$$I = \int_S \vec{J}\cdot d\vec{s}$$

One can then write a more general form of the Biot-Savart law as

$$\boxed{\vec{B}(x) = \frac{1}{c} \int d^3x' \ \vec{J}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}$$

One can verify that for a thin wire it reduces to the earlier form: Note that in general, $d^3x' = d\vec{s}\cdot d\vec{l}$. For a thin wire, $\vec{J}$ is parallel to $d\vec{l}$, so that

$$\int_V d^3x' \ \vec{J}\times(\vec{x}-\vec{x}') = \int_V \vec{J}\times(\vec{x}-\vec{x}')\, d\vec{s}\cdot d\vec{l} = \int_l d\vec{l} \times (x-\vec{x}') \int_S \vec{J}\cdot d\vec{s}$$

In the last step we have assumed that the variation of $\vec{x}'$ on S is negligiable as compared to $\vec{x}$. Then, the expression is $I\int_l dl \times (x-\vec{x}')$

Differential Equations for $\vec{B}$:

To obtain $\nabla\cdot B$ and $\nabla\times B$, note that the Biot-Savart law can be written as

$$\vec{B}(\vec{x}) = \frac{-1}{c}\int d^3x' \; \vec{J}(\vec{x}')\times\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$= \frac{1}{c}\vec{\nabla}\times\left(\int d^3x' \; \vec{J}(\vec{x}')/|\vec{x}-\vec{x}'|\right)$$

This directly implies,

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

$\Rightarrow$ absence of magnetic charges by Gauss law.

To evaluate $\vec{\nabla}\times\vec{B}$, one needs the identity,

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}.$$

$\Rightarrow$

$$\nabla\times\vec{B} = \frac{1}{c}\nabla\left(\nabla\cdot\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) - \frac{1}{c}\int d^3x' \,\vec{J}(\vec{x}')\,\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$= \frac{1}{c}\nabla\int d^3x' \; \vec{J}(\vec{x}')\cdot\nabla\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) - \frac{1}{c}\int d^3x' \; \vec{J}(\vec{x}')\left(-4\pi\delta(\vec{x}-\vec{x}')\right)$$

$$= \frac{-1}{c}\nabla\int d^3x' \; \vec{J}(\vec{x}')\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) + \frac{4\pi}{c}\vec{J}(\vec{x})$$

$$= \frac{-1}{c}\nabla\int d^3x'\Bigg[\underbrace{\nabla'\cdot\left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)}_{\oint_S d\vec{S}'\cdot\frac{\vec{J}(x')}{|\vec{x}-\vec{x}'|} = 0} - \underbrace{\frac{\nabla'\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}_{=0}\Bigg] + \frac{4\pi}{c}\vec{J}(\vec{x})$$

$\therefore$

$$\boxed{\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}}$$

The validity of this equation and $\nabla \cdot \vec{B} = 0$ extends beyond the Biot-Savart law from which they were obtained. We encountered a similar situation in electrostatics. In general, the solutions of these differential equations depend on boundary conditions. When the boundary is at infinity (i.e., the current densities and fields vanish at the boundary) then one gets the Biot-Savart law as the solution.

Ampere's law:

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot \vec{ds} = \frac{4\pi}{c} \int_S \vec{J} \cdot \vec{ds}$$

$\Rightarrow$

$$\boxed{\oint_l \vec{B} \cdot \vec{dl} = \frac{4\pi}{c} I} \qquad \text{(Ampere's law)}.$$

This is the counterpart of Gauss's law in electro statics. For a straight, infinite conductor, the set up has axial symmetry. We can then choose l to be a circle of radius r around the conductor. This leads to

$$(2\pi r) B_\varphi = \frac{4\pi}{c} I$$

or,

$$\boxed{B_\varphi(r) = \frac{2I}{cr}}$$

From the vector structure of Biot-Savart law we can see that this is the only non-vanishing component of $\vec{B}$ in the problem.

The Vector Potential

From our knowledge of vector calculus, we know that

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0 \iff \vec{B} = \vec{\nabla}\times\vec{A}}$$

for some vector field $\vec{A}$. This is called the vector potential, in analogy with the scalar potential Φ. The above equation does not define $\vec{A}$ uniquely in terms of $\vec{B}$. In fact, if $\vec{A}$ is a vector potential leading to a magnetic field $\vec{B}$, then $\vec{\tilde{A}} = A + \nabla\psi$ will also lead to the same magnetic field since $\nabla\times\nabla\psi = 0$. Here ψ is any arbitrary function. The transformation

$$\vec{A} \longrightarrow \vec{\tilde{A}} = \vec{A} + \vec{\nabla}\psi$$

is called a gauge transformation. It is the counter part of $\Phi \longrightarrow \Phi + c$ in electro statics.

In terms of $\vec{A}$ the remaining magnetostatic equation becomes

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{J}$$

The gauge freedom of $\vec{A}$ can always be used to set $\nabla\cdot\vec{A}=0$. (This means that if $\vec{A}$ is not **divergenceless**, then it can be shifted by $\nabla\psi$, for an appropriately chosen ψ, such that the new $\vec{A}$ is **divergenceless**). Then one gets :

$$\boxed{\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J}} \qquad (\nabla\cdot\vec{A}=0)$$

Thus, each component of $\vec{A}$ satisfies the Poisson equation,

$$\boxed{\nabla^2 A_i = \frac{-4\pi}{c} J_i} \qquad \text{(In Coulomb gauge)}$$

The discussion of the Poisson equation in electrostatics can be applied to each component A_i separately, replacing ρ by J_i/c $i=1,2,3$. However, not all the three components are entirely independent since, $\nabla\cdot\vec{J}=0$. Furthermore, the boundary conditions on Φ and A_i are different.

When the boundary is chosen at infinity where fields and current densities on it vanish, then, in analogy with electrostatics, the solution is

$$\boxed{A_i(\vec{x}) = \frac{1}{c}\int d^3x' \, \frac{J_i(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

It satisfies $\nabla\cdot\vec{A}=0$:

$$\nabla\cdot\vec{A}=\frac{1}{c}\int d^3x'\,\nabla\cdot\left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)=-\frac{1}{c}\int d^3x'\,\nabla'\cdot\left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)$$

$$=-\frac{1}{c}\oint_{S\to\infty}\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}=0\quad\left(\text{since }\nabla'\cdot\vec{J}(\vec{x}')=0\right)$$

The magnetic field corresponding to this $\vec{A}$ is the Bist-Savart Law,

$$\vec{B}=\nabla\times\vec{A}=\frac{-1}{c}\int d^3x'\,\vec{J}(\vec{x}')\times\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

Note again that as far as the relation to $\vec{B}$ is concerned, we could add any $\vec{\nabla}\psi$ to $\vec{A}$. However, the differential equation for $\vec{A}$ has a simple form when $\vec{\nabla}\psi$ is adjusted such that $\nabla\cdot\vec{A}=0$. This referred to a choosing the Coulomb gauge. Since the $\vec{A}$ we have obtained above is already in the Coulomb gauge, we can only add a $\nabla\psi$ to it provided $\nabla\cdot(\vec{A}+\vec{\nabla}\psi)=0$ or $\nabla^2\psi=0$. If we demand that ψ is finite everywhere, including at infinity, then the only solution is $\psi=$ constant, for which $\vec{\nabla}\psi=0$.

We can now compile a list of corresponding expressions in electrostatics and magnetostatics.

Correspondance between Electrostatics and Magnetostatics :

$\vec{E}$	$\vec{B}$
$\vec{F}_e = q\vec{E}$	$\vec{F}_m = \frac{1}{c} q \vec{v} \times \vec{B}$
$\vec{E}(\vec{x}) = \int d^3x' \rho(\vec{x}') \frac{(\vec{x}-\vec{x}')}{\lvert\vec{x}-\vec{x}'\rvert^3}$	$\vec{B} = \frac{1}{c}\int d^3x' \vec{J}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{\lvert\vec{x}-\vec{x}'\rvert^3}$
$\rho(\vec{x})$	$\vec{J}(\vec{x})/c$
$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$	$\vec{\nabla}\times\vec{B} = 4\pi\vec{J}/c$
$\left(\oint_S \vec{E}\cdot d\vec{s} = 4\pi q\right)$	$\left(\oint_l \vec{B}\cdot d\vec{l} = 4\pi I/c\right)$
$\vec{\nabla}\times\vec{E} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$
$\vec{E} = -\vec{\nabla}\Phi$	$\vec{B} = \vec{\nabla}\times\vec{A}$
$\Phi \to \tilde{\Phi} = \Phi + c$	$\vec{A} \to \tilde{\vec{A}} = \vec{A} + \vec{\nabla}\psi$
$\Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{\lvert\vec{x}-\vec{x}'\rvert}$	$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}')}{\lvert\vec{x}-\vec{x}'\rvert}$
$\nabla^2\Phi = -4\pi\rho$	$\nabla^2\vec{A} = -4\pi\vec{J}/c \quad (\vec{\nabla}\cdot\vec{A} = 0)$

More correspondances will appear as we proceed with our study of magnetostatics.

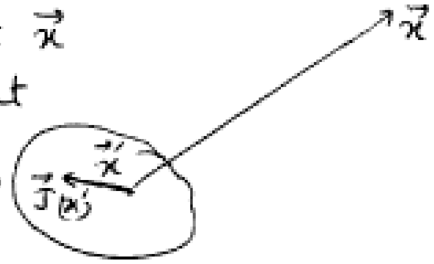
Magnetic Moment of a Localized Current Distribution :

We have found the vector potential $\vec{A}(\vec{x})$ due to a localized current distribution $\vec{J}(\vec{x})$ as

$$A_i(\vec{x}) = \frac{1}{c}\int d^3x' \; \frac{J_i(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

In analogy with electrostatics, one can expand this in terms of multipole fields or multipole moments of the current distribution $\vec{J}$. We retain only upto the dipole term.

For the observation point at $\vec{x}$ far away from the current distribution, $|\vec{x}| \gg |\vec{x}'|$ and,



$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \cdots$$

$\Rightarrow$

$$A_i(\vec{x}) = \frac{1}{c|\vec{x}|}\int d^3x' \, J_i(\vec{x}') + \frac{\vec{x}}{c|\vec{x}|^3}\cdot\int \vec{x}' \, J_i(\vec{x}')\, d^3x' + \cdots$$

In analogy with electrostatics, we can identify the magnetic monopole and dipole moments. However, $\vec{J}$ is not arbitrary as $\vec{\nabla}\cdot\vec{J} = 0$. This can be used to better understand the content of the above expansion. Consider a locali-zed $\vec{J}(\vec{x}')$ with two well behaved functions $f(\vec{x}')$ and $g(\vec{x}')$ to be specified below. Then,

$$\int d^3x' \, \nabla' \cdot (f \vec{J} g) = \oint_{S \to \infty} (f \vec{J} g) \cdot d\vec{s} = 0 \quad \text{(localization)}$$

along with $\nabla' \cdot \vec{J}(\vec{x}') = 0$, implies,

$$\int d^3x' \left[(\nabla' f) \cdot \vec{J} g + f \vec{J} \cdot (\nabla' g) \right] = 0.$$

For $f = 1$, $g = x_i'$, this reduces to

$$\boxed{\int d^3x' \; J_i(\vec{x}') = 0}$$

Hence, the magnetic monopole moment of $\vec{J}(\vec{x})$ vanishes (as expected from $\nabla \cdot B = 0$). Then,

$$\boxed{A_i(\vec{x}) = \frac{1}{c|\vec{x}|^3} \int d^3x' \; x_j x_j' \; J_i(x') + \cdots}$$

Now, choosing $f = x_i'$ and $g = x_j'$, we get

$$\int d^3x' \left(J_i x_j' + x_i' J_j \right) = 0 .$$

Using this in A_i, we get

$$A_i = \frac{1}{2c|\vec{x}|^3} \int d^3x' \sum_j x_j \left(x_j' J_i + x_j' J_i \right)$$

$$A_i(\vec{x}) = \frac{-1}{2c|\vec{x}|^3} \int d^3x' \quad x_j \left(x'_i \, J_j - x'_j \, J_i \right)$$

This can be written in a recognizable form by using the expression for the components of a vector product in terms of the totally anti-symmetric tensor ϵ_{ijk} ($\epsilon_{123} = +1$, $\epsilon_{ijk} = -\epsilon_{jik}$, $\epsilon_{ijk} = -\epsilon_{ikj}$). Consider $\vec{C} = \vec{A} \times \vec{B}$. The components $C_i = (\vec{A} \times \vec{B})_i$, A_i and B_i are related by,

$$(\vec{A} \times \vec{B})_k = \sum_{i,j} A_i B_j \epsilon_{ijk} \,, \quad \sum_k \epsilon_{ijk} (\vec{A} \times \vec{B})_k = A_i B_j - A_j B_i$$

Then,

$$A_i(\vec{x}) = \frac{-1}{2c|\vec{x}|^3} \int d^3x' \quad x_j \, \epsilon_{ijk} \left(\vec{x}' \times \vec{J} \right)_k$$

$$= \frac{-1}{2c|\vec{x}|^3} \int d^3x' \left[\vec{x} \times (\vec{x}' \times \vec{J}) \right]_i$$

Now, the magnetic (dipole) moment is defined as

$$\vec{m} = \frac{1}{2c} \int d^3x' \left(\vec{x}' \times \vec{J}(\vec{x}') \right)$$

$$= \int d^3x' \, \vec{\mathcal{M}}(\vec{x}')$$

where, $\vec{\mathcal{M}}(\vec{x}') = \frac{1}{2c} \vec{x}' \times \vec{J}(\vec{x}')$ is the magnetic moment density. In terms of $\vec{m}$,

$$\boxed{\vec{A}(x) = \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}}$$

The magnetic field $\vec{B} = \nabla \times \vec{A}$ can be evaluated as

$$\vec{B}(\vec{x}) = \frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}, \quad \hat{n} = \frac{\vec{x}}{|\vec{x}|}.$$

$$\vec{B} = \vec{\nabla} \times \left(\frac{m \times \vec{x}}{|x|^3}\right)$$

Use, $\nabla \times (\vec{a} \times \vec{b}) = \vec{a}(\nabla\cdot\vec{b}) - \vec{b}(\nabla\cdot\vec{a}) + (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$

for $\vec{a} = \vec{m}$, $\nabla\cdot\vec{a} = 0$, $(b\cdot\nabla)\vec{a} = 0$

$$\vec{\nabla} \times \left(\vec{m} \times \frac{\vec{x}}{|x|^3}\right) = \vec{m}\, \nabla\cdot\left(\frac{\vec{x}}{|x|^3}\right) - (\vec{m}\cdot\nabla)\frac{\vec{x}}{|x|^3}$$

Macroscopic form of Magnostatics Equations:

Electrons have an intrinsic magnetic moment and also their orbital motion gives rise to a further atomic magnetic moment. In the presence of an external field these atomic moments can align themselves giving rise to an average macroscopic dipole moment. The total macroscopic field is then the sum of the externally applied field and the averaged out dipole fields. The modifications of the equations can be worked out in analogy with electrostatics.

The non-existance of magnetic monopoles is not affected by the presence of the medium and hence the macroscopic $\vec{B}$ still satisfies,

$$\vec{\nabla}\cdot\vec{B} = 0$$

In turn, this implies that $\vec{B}$ can still be derived from a macroscopic vector potential $\vec{A}$. Let us denote the indevidual atomic magnetic moments by $\vec{m}_a$ $a = 1, \ldots, N_{\Delta V}$, for atoms in a volume ΔV. Then the average moment per atom is

$$\langle\vec{m}\rangle_{\Delta V} = \frac{1}{N_{\Delta V}} \sum_{a=1}^{N_{\Delta V}} \vec{m}_a$$

One can now define a magnetic moment density $\vec{M}(\vec{x})$ in the limit $\Delta V \to 0$,

$$\vec{M}(\vec{x}) = N \lim_{\Delta V \to 0} \langle m \rangle_{\Delta V}$$

where N is the **local** no. density of atoms.

Now consider a volume ΔV with current density $\vec{J}(\vec{x}')$ and moment density $\vec{M}(\vec{x}')$, where $\vec{x}'$ is a point within ΔV. These cont- **ribute** to the potential at $\vec{x}$, as

$$\delta \vec{A}(\vec{x}) = \frac{1}{c} \frac{\vec{J}(\vec{x}')\, \Delta V}{|\vec{x}-\vec{x}'|} + \frac{\vec{M}(\vec{x}') \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \Delta V$$

and,

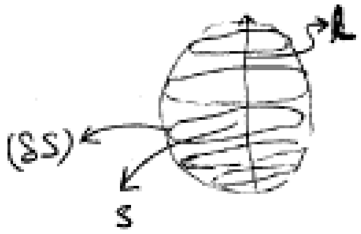
$$\vec{A}(\vec{x}) = \frac{1}{c} \int d^3x' \left(\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + c\vec{M}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \right)$$

Note that,

$$\int d^3x' \ \vec{M}(\vec{x}') \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \int d^3x' \ \vec{M}(\vec{x}') \times \nabla' \left(\frac{1}{|\vec{x}-\vec{x}'|} \right)$$

$$= -\int d^3x' \ \nabla' \times \left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right) + \int d^3x' \ \frac{\nabla' \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

The first term has only boundary contributions and vanishes if $\vec{M}$ is localized:



$$\int d^3x' \ \nabla' \times (\vec{\ }) = \int_L \int_S d\vec{l} \cdot d\vec{S} \ \nabla' \times (\vec{\ })$$

$$= \int_l d\vec{l} \int_{\delta S} d(\delta \vec{S}) \cdot \left(\frac{\vec{M}(x')}{|\vec{x}-\vec{x}'|} \right) = 0$$

Here, the surface S sweeps the volume V as it is moved along l. δS is the boundary of S and $d(\delta\vec{S})$ is the element of length along the boundary of S.

Then,

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3x' \; \frac{\vec{J}(\vec{x}') + c\,\vec{\nabla}' \times \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

From this it is clear that $c\vec{\nabla}\times\vec{M}$ can be regarded as a magnetization current much in the same way as polarization charge density in electrostatics. Regarding $\vec{J} + c\nabla\times M$ as the modified current, the modified equation for $\vec{B}$ is obtained as

$$\boxed{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\left(\vec{J}(x) + c\,\vec{\nabla}\times\vec{M}(\vec{x})\right)}$$

or

$$\vec{\nabla}\times(\vec{B} - 4\pi\vec{M}) = \frac{4\pi}{c}\vec{J}(x)$$

If we define $\vec{H} = \vec{B} - 4\pi\vec{M}$, then,

$$\boxed{\vec{\nabla}\times\vec{H}(\vec{x}) = \frac{4\pi}{c}\vec{J}(\vec{x})}$$

$\vec{H}$ is the analogue of $\vec{D}$ in electrostatics.

$(\vec{H}, \vec{D})$ vs $(\vec{B}, \vec{E})$

It should be emphasized that $\vec{B}$ and $\vec{E}$ are always the physical fields. These determine the macroscopic force on a stationary or moving charge through $f_e = q\vec{E}$ and $\vec{F}_m = \frac{q}{c}\vec{v}\times\vec{B}$. $\vec{H}$ and $\vec{D}$ are introduced to write the eqn. in a simple form. The modified equations also show that by changing the external parameters ρ, $\vec{J}$ we can directly change $\vec{H}$ and $\vec{D}$ and have them fully under control. However, $\vec{B}$ and $\vec{E}$ are also affected by the responce of the medium and hence, for an unknown medium, are not fully under our control.

Relation between $\vec{H}$ and $\vec{B}$

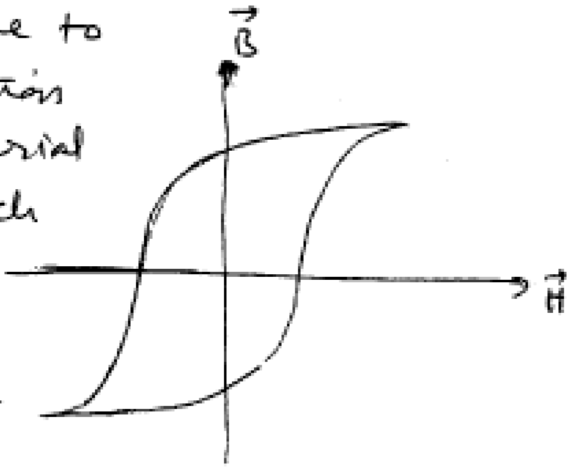
The magnetic properties of material are usually more complicated than their electric properties. For some materials (diamagnetic and paramagnetic) one can write a linear relationship between $\vec{B}$ and $\vec{H}$:

$$\boxed{\vec{B} = \mu \vec{H}}$$

μ is called the magnetic permeability. For paramagnetic substances (which are weakly attracted by a magnetic field) $\mu > 1$ and for diamagnetic substances (which are weakly repelled by a magnetic field) $\mu < 1$. For ferromagnetic substances which are very strongly affected by the applied field, the relationship between $\vec{B}$ and $\vec{H}$ is non-linear

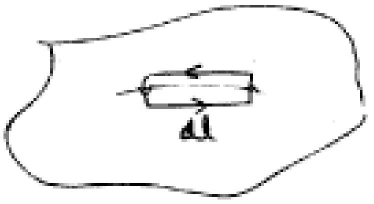
$$\vec{B} = F(\vec{H})$$

In fact, $\vec{B}$ is not a single valued function of $\vec{H}$ which is caused by the phenomenon of hysteresis. This is due to the finite relaxation time in the material as a result of which the variation in $\vec{B}$ does not instantaneously follow the variation in $\vec{H}$ or equivalently the variation in the externally controlled currents.



Boundary conditions across a current sheet:

$$(\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0.$$



$$\hat{n}\times(\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c}\,\vec{K}$$

$\vec{K}$: surface current density on the sheet, i.e., surface current per unit length

Force and Energy for a Localized Current Distribution in an External Magnetic Field

Let us start with the Lorentz force,

$$F = \frac{q}{c} \vec{V} \times \vec{B}$$

For a continuous charge distribution, the force on a volume ΔV is

$$\Delta \vec{F} = \frac{\rho \vec{V}}{c} \times \vec{B} \, \Delta V$$

Hence, the force on the whole current distribution is

$$\vec{F} = \frac{1}{c} \int \vec{J} \times \vec{B}(x) \, d^3x .$$

or

$$F_i = \frac{1}{c} \int \epsilon_{ijk} J_j B_k \, d^3x.$$

Let us assume that $\vec{B}$ varies lowly over the region where $\vec{J}$ is non-zero. Then we can expand it around some $\vec{x} = 0$ within the distribution as

$$B_k(x) = B_k(0) + \vec{x} \cdot \vec{\nabla} B_k \Big|_{x=0} + \cdots$$

so that,

$$F_i = \frac{1}{c} \epsilon_{ijk} B_k(0) \overset{=0}{\int d^3x \, J_j} + \frac{1}{c} \int d^3x \, \epsilon_{ijk} J_j \, \vec{x} \cdot \vec{\nabla} B_k(0) + \cdots$$

or

$$F_i = \frac{1}{c}\epsilon_{ijk}\left(\int d^3x \;\; J_j\, x_l\right)\; \nabla_l B_k(0)$$

$$= \frac{1}{2c}\epsilon_{ijk}\left[\int d^3x\left(J_j\, x_l - J_l x_j\right)\right]\nabla_l B_k\Big|_{x=0}$$

$$= \frac{1}{2c}\;\epsilon_{ijk}\left[\int d^3x\;\; \epsilon_{ljn}\left(\vec{x}\times\vec{J}\right)_n\right]\nabla_l B_k\Big|_{x=0}$$

$$= \;\; \epsilon_{ijk}\;\; \epsilon_{ljn}\,(\vec{m})_n\;\; \nabla_l B_k\Big|_{x=0}$$

$$= \;\; \epsilon_{ijk}\;\; \left(\vec{m}\times\vec{\nabla}\right)_j\; B_k$$

$$= \left[\left(\vec{m}\times\vec{\nabla}\right)\times\vec{B}\;\right]_i$$

Therefore,

$$\vec{F} = \left(\vec{m}\times\vec{\nabla}\right)\times\vec{B}$$

$$= \nabla\left(\vec{m}\cdot\vec{B}\right) - \vec{m}\,\overbrace{\left(\vec{\nabla}\cdot\vec{B}\right)}^{=0}$$

Hence and

$$\boxed{\begin{aligned}\vec{F} &= \nabla\left(\vec{m}\cdot\vec{B}\right)\\ U &= -\,\vec{m}\cdot\vec{B}\end{aligned}}$$

where U is the potential energy, $\vec{F} = -\vec{\nabla}U$.

Note that $-\vec{m}\cdot\vec{B}$ is the energy of the magnetic moment $\vec{m}$ alone, and not that of the moment + field system. In bringing $\vec{m}$ to its position and orientation, work has to be done to keep $\vec{B}$ constant, countering the backreaction of $\vec{m}$ on $\vec{B}$. U does not include this.