Electrostatics in Dielectric Media

- Polarization and Polarization Charge Density
- Electric Displacement
- Modification of the Electrostatic Equations
- Susceptibility and Permittivity
- Boundary Conditions at Interfaces
- An Application of the Method of Images to Dielectric Media

(Scanned class notes – S. F. Hassan)

Electrostatics with Ponderable Media.

So far we have considered electric fields in media that do not respond to it, like vacuum or air. Then there is no distinction between microscopic and macroscopic fields. However matter is made up of charges that get polarized under the influence of an applied elecrostatic field. Then, since positive and negative charges do not cancel at the microscopic level, various multipole fields are set up inside matter in response to an applied external field. The total (net) field inside matter then is the sum of the applied fields and the microscopic fields developed in response to it. For macroscopic measurements, one is not interested in the microscopic details of the induced fields, but only in their macroscopic average. For most media, only the average dipole moment of the polarization is significant and below we concentrate only on this case.

We will now describe how $\vec{\nabla}\times\vec{E}=0$ and $\vec{\nabla}\cdot\vec{E}=4\pi\rho$ get modified as a result of the responce of the medium. From now on, $\vec{E}$ will denote the net electric field, including the medium response. The curl equation describes a basic property of the electric field which is the path independence of the work done by it. This holds for the external fields, as well as molecular fields, and hence for the sum. Therefore, even inside a dielectric medium,

$$\boxed{\vec{\nabla} \times \vec{E} = 0}$$

The implication is that the macroscopic field inside the medium is also derivable from a potential.

The external field causes charge separation inside molecules, making them behave like tiny electric dipoles. We are interested in the average macroscopic behaviour of these dipoles. Consider a volume ΔV containing $N_{\Delta V}$ molecules with dipole moments p_a, $a = 1, \ldots, N_{\Delta V}$. Then, the average dipole moment/molecule is

$$\langle p \rangle = \frac{1}{N_{\Delta V}} \left(\vec{p}_1 + \vec{p}_2 + \cdots + p_{N_{\Delta V}} \right)$$

Now, if N is the local number density of molecules in a medium, then the average polarization per unit volume (for a small volume) is given by

$$\vec{P} = N \langle p \rangle$$

If the medium is composed of several species "i" of molecules, then,

$$\boxed{\vec{P} = \sum_i N_i \langle p \rangle_i}$$

Note that as is standard, we have taken ΔV to by large on molecular scales but small on macroscopic scales.

The total charge density consists of net charges carried by the molecules (if they are charged) and excess free charges in the medium.

Now, we can find the electrostatic potential in such a medium. Consider a small volume ΔV around a point $\vec{x}'$ in the medium. It contains a charge $\rho(x')\Delta V$ and a dipole moment $\vec{P}\Delta V$. Consider a point $\vec{x}$ outside ΔV. The potential at $\vec{x}$ due to ΔV is

$$\Delta \Phi(\vec{x}, \vec{x}') = \frac{\rho(x')}{|\vec{x} - \vec{x}'|}\Delta V + \frac{\vec{P}(x') \cdot (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}\Delta V$$

Then the total potential at $\vec{x}$ is given by.

$$\Phi(\vec{x}) = \int d^3x' \; \frac{\rho(x')}{|\vec{x} - \vec{x}'|} + P(x') \cdot \nabla'\left(\frac{1}{|\vec{x} - \vec{x}'|^3}\right)$$

Integrating by parts and dropping a surface term,

$$\boxed{\Phi(\vec{x}) = \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|}\left(\rho(x') - \vec{\nabla}' \cdot \vec{P}(x')\right)}$$

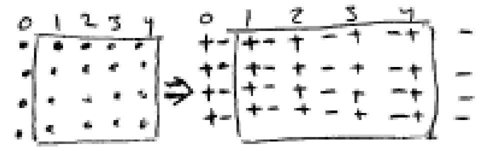
Using $\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$, one obtains the modified Maxwell's equation as

$$\nabla\cdot\vec{E} = -\nabla^2\Phi = \int d^3x' \; 4\pi\,\delta(\vec{x}-\vec{x}')\left(\rho(\vec{x}') - \nabla'\cdot\vec{P}(\vec{x}')\right)$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = 4\pi\left(\rho - \vec{\nabla}\cdot\vec{P}\right)}$$

This indicates that $\nabla\cdot\vec{P}$ behaves as an additional charge density, the charge density created as a result of polarization, ρ_{pol}.

$$\boxed{\vec{\nabla}\cdot\vec{P} = -\rho_{pol}}$$



If $\vec{P}$ is constant (in space), then $\rho_{pol}=0$ (equal no. of charges enter and leave a small region of space). But if the polarization varies in space, then ~~net~~ net charges could get accumolated in ΔV due to a polarization gradient across it. This is specially the case at the boundaries where the electric properties of the medium changes.

$$\vec{\nabla}\cdot\vec{E} = 4\pi\underbrace{\left(\rho + \rho_{pol}\right)}$$

(net charge density measured by the Gauss's law)

Electric Displacement $\vec{D}$:

The modified Maxwell equation has the form

$$\nabla \cdot (\vec{E} + 4\pi \vec{P}) = 4\pi \rho$$

Define: $\vec{D} = \vec{E} + 4\pi \vec{P}$ (Electric Displacement)

Then, $\boxed{\vec{\nabla} \cdot \vec{D} = 4\pi \rho}$

$\vec{D}$ can be regarded as the electric field that ρ would have generated in the free space, i.e, in the absence of the dielectric medium. Hence, $4\pi \vec{P}$ is the difference between the would-have-been field $\vec{D}$ and the actual field $\vec{E}$.

Relation between $\vec{D}$ and $\vec{E}$:

The polarization $\vec{P}$ is caused by the applied field, the measurable part of which is $\vec{E}$. (It is either due to charge separation caused by $\vec{E}$, or due to reorientation of already existing molecular dipole moments) To find the relation between $\vec{E}$, $\vec{P}$ and $\vec{D}$, we make some simplifying assumptions about the medium. In practice, these assumptions are generally valid for most dielectric media at moderate values of $\vec{E}$.

1) Linear response: $\vec{P} \propto \vec{E}$: $P_i = \chi_{ij} E_j$

2) Isotropy: $\chi_{ij} = \chi_e \delta_{ij} \Rightarrow$

$$\boxed{\vec{P} = \chi_e \vec{E}}$$

χ_e is called electric susceptibility.

Then, $\vec{D} = \vec{E} + 4\pi \vec{P} = (1 + 4\pi\chi_e)\vec{E}$

$$\boxed{\vec{D} = \epsilon \vec{E}}$$

$\boxed{\epsilon = 1 + 4\pi\chi_e}$: dielectric constant, (relative permittivity).

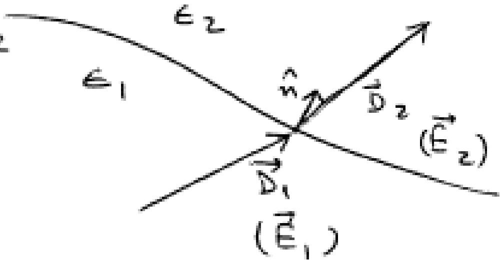
3) $\nabla \cdot (\epsilon \vec{E}) = 4\pi\rho$.

In general, $\vec{\nabla}\epsilon \neq 0$. But if the medium is uniform, then $\vec{\nabla}\epsilon = 0$ and $\epsilon \vec{\nabla} \cdot \vec{E} = 4\pi\rho$.

$$\boxed{\vec{\nabla} \cdot \vec{E} = 4\pi(\rho/\epsilon)}$$

Thus inside such media the electric field behaves as in free space corresponding to a reduced charge ρ/ϵ. (~~what~~ note that $\epsilon = 1 + 4\pi\chi > 1$). The difference arises if media of different electrical properties (different dielectric constants) are juxtaposed.

Consider the boundary between two media of dielectric constants ϵ_1 and ϵ_2 and a surface charge density σ for free (as opposed to bound) charges.
Then, the same derivation as for the free space case leads to



$$\boxed{\begin{aligned} (\vec{D}_2 - \vec{D}_1)\cdot\hat{n} &= 4\pi\sigma \\ (\vec{E}_2 - \vec{E}_1)\cdot\hat{t} &= 0 \end{aligned}}$$

The equation for the tangential components has the same content as before. But the equation for the normal components has a further implication. consider the case of $\sigma = 0$, i.e., no free charges on the boundary. Then, while $\vec{D}$ is continuous across the boundary, $\vec{E}$ is not:

$$(\epsilon_2 \vec{E}_2 - \epsilon_1 \vec{E}_1)\cdot\hat{n} = 0 \quad \Rightarrow (\vec{E}_2 - \vec{E}_1)\cdot\hat{n} \neq 0$$

This discontinuity is a result of polarization charges that develop at the boundary due to different amounts of charge separation on the two sides.

In fact it can be easily verified that

$$\boxed{(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\, \sigma_{pol.}}$$

(show this. Note that the polarization charge density is given by

$$\boxed{\vec{\nabla} \cdot \vec{P} = -\rho_{pol}}$$

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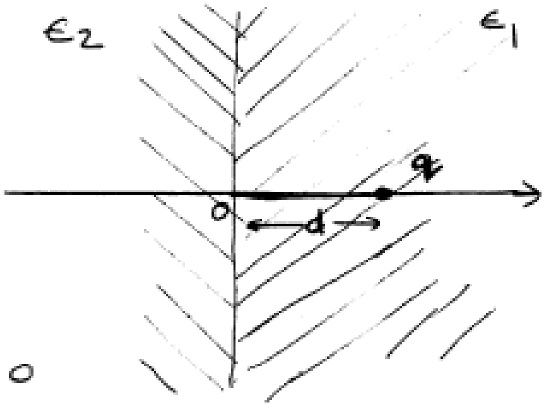
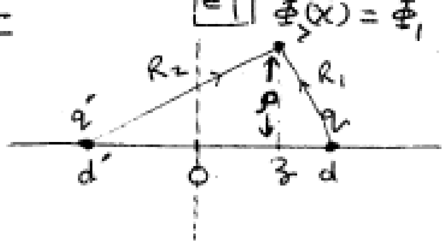
An Application of the Method of Images to Dielectric Media:

Consider two media of dielectric constants ϵ_1 and ϵ_2 separated by a boundary at $z = 0$. A point charge q is placed at $z = d > 0$ in the medium with dielectric constant ϵ_1. We want to evaluate the fields inside both media i.e., for both $z > 0$ and $z < 0$.

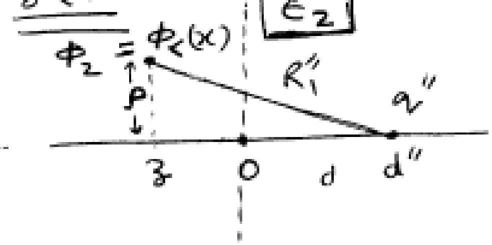


Physically, the charge q gives rise to a polarization charge density on the boundary at $z = 0$. Both q and σ_{pol} contribute to the value of the potential. The problem can be solved by finding appropriate image charges that reproduce the effects of the polarization charges.

$z > 0$



$z < 0$



Let us choose cylindrical coordinates around the z-axis so that the observation point $\vec{x}$ has coordinates (ρ, z, ϕ). Due to azimuthal symmetry the angle ϕ will not appear anywhere.

When the observation point has $z>0$, we introduce an image charge q' at d' and regard the entire medium as having dielectric const. ϵ_1 (the dielectric constant of the medium in which the observation is made). Then,

$$\Phi_{(1)} = \frac{1}{\epsilon_1}\left(\frac{q}{|R_1|} + \frac{q'}{|R_2|}\right).$$

$$\left(R_1 = \sqrt{\rho^2 + (d-z)^2}\ , \qquad R_2 = \sqrt{\rho^2 + (d'+z)^2} \right)$$

where q' and d' are to be determined.

When the observation point has $z<0$, we replace q by a charge q'' placed at d'' in a medium of dielectric constant ϵ_2. Then,

$$\Phi_{(2)} = \frac{1}{\epsilon_2}\frac{q''}{|R_1''|}\ , \quad |R_1''| = \sqrt{\rho^2 + (d''-z)^2}$$

(**Note that now** $z<0$)

q'' and d'' are to be determined. The parameters q', q'', d', d'' are determined by the boundary conditions at $z=0$. These are (as we have seen),

$$\boxed{\begin{aligned} \epsilon_1 E_{(1)z} &= \epsilon_2 E_{(2)z} \quad (\text{at } z=0) \\ E_{(1)\rho} &= E_{(2)\rho} \quad (\text{at } z=0) \end{aligned}}$$

Here, E_z and E_ρ denote the normal and tangential components of the field, respectively (the other tangential component E_φ vanishes). $E_{(a)z}$ and $E_{(a)\rho}$ (for $a=1,2$) are given by

$$E_{(a)z} = -\frac{\partial}{\partial z}\Phi_{(a)} \qquad E_{(a)\rho} = -\frac{\partial}{\partial \rho}\Phi_{(a)}.$$

Now,

$$\frac{\partial}{\partial z}(R_1)^{-1}\bigg|_{z=0} = \frac{d}{(\rho^2+d^2)^{3/2}}, \quad \frac{\partial}{\partial z}(R_1'')^{-1}\bigg|_{z=0} = \frac{d''}{(\rho^2+d''^2)^{3/2}}$$

$$\frac{\partial}{\partial z}(R_2)^{-1}\bigg|_{z=0} = -\frac{d'}{(\rho^2+d'^2)^{3/2}}$$

$$\therefore \quad \epsilon_1 \frac{\partial}{\partial z}\Phi_1\bigg|_{z=0} = \epsilon_2 \frac{\partial}{\partial z}\Phi_2\bigg|_{z=0}$$

$$\Rightarrow \quad q\frac{d}{(\rho^2+d^2)^{3/2}} - q'\frac{d'}{(\rho^2+d'^2)^{3/2}} = q''\frac{d''}{(\rho^2+d''^2)^{3/2}}$$

Note that the magnitudes and locations of the image charges should be independent of the location of the observation point, and hence ρ. This is possible if

$$\boxed{d = |d'| = d''}$$

Then,

$$\boxed{q - q' = q''}$$

For matching the tangential components we need,

$$\frac{\partial}{\partial \rho}\left(\frac{1}{|R_1|}\right)\bigg|_{z=0} = \frac{\partial}{\partial \rho}\left(\frac{1}{|R_2|}\right)\bigg|_{z=0} = \frac{-\rho}{(\rho^2+d^2)^{3/2}}$$

Then, $\frac{\partial}{\partial \rho}\Phi_1\Big|_{z=0} = \frac{\partial}{\partial \rho}\Phi_2\Big|_{z=0}$ implies,

$$\boxed{\frac{q}{\epsilon_1} + \frac{q'}{\epsilon_1} = \frac{q''}{\epsilon_2}}$$

q' and q'' can now be determined as

$$\boxed{\begin{aligned} q' &= -\left(\frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}\right)q \\ q'' &= \left(\frac{2\epsilon_2}{\epsilon_2+\epsilon_1}\right)q \end{aligned}}$$

The polarization surface charge density can be easily computed from the discontinuity in E_z:

$$\sigma_{pol} = \frac{1}{4\pi}\left(E_{(2)z} - E_{(1)z}\right) = \frac{1}{4\pi}\left[\frac{q'' d\, \epsilon_2^{-1}}{(\rho^2+d^2)^{3/2}} - \right.$$

$$= \frac{1}{4\pi}\left[\frac{q''}{\epsilon_2}\frac{d}{(\rho^2+d^2)^{3/2}} - \frac{q}{\epsilon_1}\frac{d}{(\rho^2+d^2)^{3/2}} + \frac{q'}{\epsilon_1}\frac{d}{(\rho^2+d^2)^{3/2}}\right]$$

$$= \frac{1}{4\pi}\left[\frac{q''}{\epsilon_2} - \frac{1}{\epsilon_1}(q-q')\right]\frac{d}{(\rho^2+d^2)^{3/2}}$$

$$\sigma_{pol} = \frac{1}{4\pi} \, q'' \left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right) \frac{d}{(\rho^2 + d^2)^{3/2}}$$

$$= \frac{q}{4\pi} \, \frac{2\epsilon_2}{(\epsilon_2 + \epsilon_1)} \, \frac{\epsilon_1 - \epsilon_2}{\epsilon_2 \epsilon_1} \, \frac{d}{(\rho^2 + d^2)^{3/2}}$$

$$\boxed{\sigma_{pol} = -\frac{q}{2\pi} \, \frac{1}{\epsilon_1} \left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right) \frac{d}{(\rho^2 + d^2)^{3/2}}}$$