Solving the Laplace Equation

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(Class notes – S. F. Hassan)

4 Solving the Laplace Equation

We now discuss a method of solving the Laplace equation based on separation of variables. First we have to understand the notion of expansion in orthogonal functions. Then we illustrate the method of separation of variables for the Laplace equation in rectangular coordinates. After that we tackle the actual problem of interest which is the Laplace equation in spherical coordinates, and the associated set of special functions.

4.1 Expansions in Orthogonal Functions

Consider a variable ξ over an interval (a, b). Also consider a set of complex or real functions $U_n(\xi)$ for, say, $n = 1, 2, \cdots$ over this interval. These functions are said to form an *orthogonal* set over the interval if

$$\int_a^b d\xi U_n^*(\xi) U_m(\xi) = 0\,, \quad \text{for} \quad n \neq m$$

Then the functions can always be normalized to unity so that

$$\boxed{\int_a^b d\xi U_n^*(\xi) U_m(\xi) = \delta_{nm}} \tag{1}$$

Furthermore, the set is said to be *complete* if

$$\boxed{\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi' - \xi)} \tag{2}$$

If such a set is given, then any function $f(\xi)$ over the interval (a, b) can be expanded in the basis of this *orthonrmal* and *complete* set of functions, much in the same way that a vector can be expanded in an orthogonal basis:

$$f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi)$$

The orthogonality allows us to determine the coefficients a_n as

$$a_n = \int_a^b d\xi U_n^*(\xi) f(\xi)$$

and completeness insures the self consistency of the expansion,

$$f(\xi) = \sum_{n=1}^{\infty} \left(\int_a^b d\xi' U_n^*(\xi') f(\xi') \right) U_n(\xi) = \int_a^b d\xi' f(\xi') \delta(\xi' - \xi)$$

The above can be straightforwardly generalized to more than one variable, for example,

$$f(\xi,\eta) = \sum_{n=1}^{\infty}\sum_{\mu=1}^{\infty} a_{n\mu} U_n(\xi) V_\mu(\eta)\,, \qquad a_{n\mu} = \int_a^b d\xi \int_c^d d\eta U_n^*(\xi) V_\mu^*(\eta) f(\xi,\eta)$$

In physics, the functions $U_n(\xi)$ usually appear as solutions of certain differential equations. In general, $f(\xi)$ is any arbitrary function unrelated to this differential equation. However, sometimes $f(\xi)$ may also satisfy a related differential equation in which case the expansion is more constrained.

The most common example of the above is expansion in a Fourier series. In this case,

$$U_n(x) = \frac{1}{\sqrt{a}} e^{i2\pi nx/a}$$

with $n = 0, \pm 1, \pm 2, \cdots$ over $-a/2 \le x \le a/2$. These are solutions to $\partial^2 U(x)/\partial x^2 = -k^2 U(x)$ for $k^2 = (2\pi n/a)^2$. The orthogonality and completeness relations then take the form

$$\frac{1}{a}\int_{-a/2}^{a/2} dx e^{i2\pi(m-m')x/a} = \delta_{mm'}\,, \qquad \frac{1}{a}\sum_{m=-\infty}^{+\infty} e^{i2\pi m(x-x')/a} = \delta(x-x')$$

These, that can be verified, also serve as very useful representations of the Kroneker and Dirac delta-functions.

4.2 Separation of Variables: Laplace Equation in Rectangular Coordinates

The method of separation of variables allows us to break a single partial differential equaiton in several variables into several differential equations each in a single variable. The full solution then has the form of an expansion in orthogonal functions discussed above. The simplest example of this procedure is the Laplace equation in rectangular coordinates,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{3}$$

Assume that a solution has the product form

$$\Phi(x,z,y) = X(x)Y(y)Z(z) \tag{4}$$

A priori solutions of this sort seem very sepecial. However, they can be used as building blocks, along with expansion in orthogonal functions, to construct the most general solution of the equation. With this factorization, the Laplace equation takes the form (after dividing the entire equation by XYZ),

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z(z)}\frac{\partial^2 Z}{\partial z^2} = 0$$

Each term on the left hand side is a function of an independent variable and in principle could be varied without affecting the other two terms. Since this is not consistent with the equation, it follows that each term should be separately constant,

$$\begin{aligned}
\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} &= -\alpha^2 \quad &\Rightarrow \quad X = e^{\pm i\alpha x} \\
\frac{1}{Y(y)}\frac{\partial^2 Y}{\partial y^2} &= -\beta^2 \quad &\Rightarrow \quad Y = e^{\pm i\beta y} \\
\frac{1}{Z(z)}\frac{\partial^2 Z}{\partial z^2} &= \gamma^2 \quad &\Rightarrow \quad Z = e^{\pm \gamma z}
\end{aligned}$$

with,

$$\alpha^2 + \beta^2 = \gamma^2$$

We have chosen α and β to be real. Now, Φ can be constructed as a product of these three factors. Since this solves the Laplace equation for any value of α and β, we could also sum over these values. So the general solution is

$$\boxed{\Phi(x,y,z) = \sum_{\alpha=-\infty}^{\infty}\sum_{\beta=-\infty}^{\infty} e^{i\alpha x}e^{i\beta y}\left(\bar{A}_{\alpha\beta+}e^{\sqrt{\alpha^2+\beta^2}z} + \bar{A}_{\alpha\beta-}e^{-\sqrt{\alpha^2+\beta^2}z}\right)} \tag{5}$$

where $\bar{A}_{\alpha\beta\pm}$ are arbitrary constants to be fixed by the boundary conditions. The summations could be either discrete or continuous, depending on the nature of their indices. One can see that this solution has the form of an expansion in orthogonal functions (in this case a Fourier expansion). The orthogonal functions emerge as solutions of the equations that arise in the process of separation of variables.

To specify the solution any further, we have to determine the $\bar{A}_{\alpha\beta\pm}$ by imposing boundary conditions. Let Φ be the potential inside a rectangular box bounded by the coordinate planes, and the three planes perpendicular to the coordinate axes at $x = a$, $y = b$ and $z = c$. We impose the boundary condition that Φ vanishes on all sides of the box except for the face at $z = c$. On this face it is given by $\Phi(x, y, z = c) = V(x, y)$. Now $\Phi = 0$ at $x = 0$ for all values of y and z implies that

$$\sum_{\alpha=-\infty}^{\infty} \bar{A}_{\alpha\beta\pm}e^{\pm\sqrt{\alpha^2+\beta^2}z} = 0 \qquad \Rightarrow \qquad \sum_{\alpha\geq 0}^{\infty}\left(\bar{A}_{\alpha\beta\pm} + \bar{A}_{-\alpha\beta\pm}\right)e^{\pm\sqrt{\alpha^2+\beta^2}z} = 0$$

which has the solution $\bar{A}_{-\alpha} = -\bar{A}_{\alpha}$ and $\bar{A}_0 = 0$, where the remaining indices have been suppressed. As a result one can write

$$\sum_{\alpha=-\infty}^{\infty}\bar{A}_{\alpha}e^{i\alpha x} = \sum_{\alpha>0}^{\infty}(2i\bar{A}_{\alpha})\sin(\alpha x)$$

The same condition applied at $y = 0$ and $z = 0$ finally leads to

$$\Phi(x,y,z) = \sum_{\alpha>0}\sum_{\beta>0} A_{\alpha\beta}\sin(\alpha x)\sin(\beta y)\sinh(\sqrt{\alpha^2+\beta^2}z)$$

The boundary conditions $\Phi(x=a,y,z)=0$ and $\Phi(x,y=b,z)=0$ give

$$\left.\begin{array}{l}\alpha_n = \dfrac{n\pi}{a} \\ \beta_m = \dfrac{m\pi}{b}\end{array}\right\} \qquad \Rightarrow \qquad \gamma_{nm} = \pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}$$

so that,

$$\boxed{\Phi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)}$$

Finally, the boundary condition $\Phi(x,y,z=c)=V(x,y)$ lets us determine the coefficients A_{nm} as

$$\boxed{A_{nm} = \frac{4}{ab\sinh(\gamma_{nm}c)}\int_0^a dx\int_0^b dy V(x,y)\sin(\alpha_n x)\sin(\beta_m y)}$$

This completely fixes the solution.

4.3 Laplace Equation in Spherical Coordinates

Let x,y,z and r,θ,ϕ denote the coordinates of the same point in Rectangular and Spherical coordinate systems, respectively. They are related by

$$z = r\cos\theta\,, x = r\sin\theta\cos\phi\,, y = r\sin\theta\sin\phi$$

Using this, the Laplace equation in the spherical coordinate system is found to be

$$\boxed{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0} \tag{6}$$

An easy way of obtaining this is to use the general expression for the Laplacian in curved space in terms of the metric g_{ij},

$$\nabla^2\Phi = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}g^{ij}\partial_j\Phi\right) = 0$$

The infinitesimal distance in spherical coordinates is given by

$$ds^2 \equiv g_{ij}d\xi^i d\xi^j = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

The non-zero components of the metric can be read off from this as $g_{rr}=1\,, g_{\theta\theta} = r^2\,, g_{\phi\phi} = r^2\sin^2\theta$. Substituting in the general expression, this leads to the Laplacian in (r,θ,ϕ) as given above.

Separation of Variables: To solve this, we again use the method of separation of variables and assume a product form for the solution,

$$\Phi(r,\theta,\phi) = \frac{U(r)}{r}P(\theta)Q(\phi)$$

Substituting in (6) and multiplying the equation by $r^2 \sin^2\theta / UPQ$ yields,

$$r^2 \sin^2\theta \left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{r^2\sin\theta}\frac{1}{P}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta}) \right] + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0$$

This can be separated into three equations in terms of r, θ and ϕ. The procedure also introduces two real constants m and l,

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \tag{7}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta})\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right] P = 0 \tag{8}$$

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0 \tag{9}$$

Solutions: The ϕ and r equations have pretty simple solutions. The complication is hidden in the θ equation which will be discussed in more detail in the next subsection. Equation (7) has the solution

$$\boxed{Q(\phi) = e^{im\phi}} \tag{10}$$

If the full azimuthal range is allowed (*i.e.*, $0 \le \phi \le 2\pi$) then the single valuedness of the solution, $Q(\phi) = Q(\phi + 2\pi)$, restricts m to be an integer. We already know that these functions form a complete orthogonal set.

The radial equation (9) has the simple solution

$$\boxed{U(r) = A_l\, r^{l+1} + B_l\, r^{-l}} \tag{11}$$

A_l and B_l are arbitrary constants. In particular, if the solution is required to remain finite at $r = 0$ (or $r \to \infty$) for say $l > 0$, then we must have $B_l = 0$ (or $A_l = 0$).

The θ-equation (8) is known as the *generalized Legendre equation* and its solutions are known as the *associated Legendre polynomials*, denoted by

$$\boxed{P(\theta) = P_l^m(\cos\theta)} \tag{12}$$

These are well defined over the range $-1 \le \cos\theta \le +1$ provided $l = 0, 1, \cdots \infty$ and, for each l, the range of m is restricted to $m = -l, -l+1, \cdots, -1, 0, +1, \cdots, l-1, l$. $P_l^m(\cos\theta)$ will be discussed in a little more detail later.

Combining the three factors, the full solution Φ can be written as the expansion

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(\bar{A}_{lm}\, r^l + \bar{B}_{lm}\, r^{-l-1}\right) P_l^m(\cos\theta) e^{im\phi} \tag{13}$$

It is customary to combine the two angular factors into new functions called *spherical harmonics*,

$$Y_{lm}(\theta,\phi) = (normalization) \times\ P_l^m(\cos\theta)e^{im\phi}$$

Then,

$$\boxed{\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}\,r^{l} + B_{lm}\,r^{-l-1}\right)Y_{lm}(\theta,\phi)} \qquad (14)$$

The $Y_{lm}(\theta, \phi)$ form a complete orthonormal set on the surface of the unit sphere which is spanned by $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. This allows one to determine A_{lm} and B_{lm} in terms of boundary conditions on Φ specified, say, on surfaces at fixed r. Since there are two sets of constants, one needs boundary conditions specified on two surfaces one of which could be at the origin or at infinity.

4.4 Legendre and Associated Legendre Polynomials

Legendre Polynomials: Let us first consider the θ-equation for $m = 0$. In terms of $x = \cos\theta$ ($dx = -\sin\theta d\theta$, $d/dx = -(1/\sin\theta)d/d\theta$), this becomes the Legendre equation,

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0$$

The solutions are the Legendre polynomial $P_l(x) = P_l(\cos\theta)$. These are given by the *Rodrigue's formula*,

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

For some small orders,

$$\begin{aligned}
&P_0(x) = 1,\\
&P_1(x) = x,\\
&P_2(x) = \tfrac{1}{2}(3x^2 - 1),\\
&P_3(x) = \tfrac{1}{2}(5x^3 - 3x),\\
&P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3),\ \cdots
\end{aligned}$$

For any l they can be computed using the recurrence relation

$$(l+1)P_{l+1} - (2l+1)xP_l + lP_{l-1} = 0$$

These polynomials satisfy the orthogonality relation

$$\int_{-1}^{+1} P_{l'}(x)P_l(x)dx = \frac{2}{2l+1}\delta_{l'l}$$

Hence the orthnormal basis is

$$U_l(x) = \sqrt{\frac{2+1}{2}}P_l(x)$$

Note that the orthogonality relation involves integrating over $x = \cos\theta$. In terms of θ,

$$\int_{-1}^{+1} dx = \int_0^{\pi} \sin\theta d\theta$$

Associated Legendre Polynomials: Let us now turn to the θ-equation for $m \neq 0$. In term of $x = \cos\theta$, this is

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] P = 0$$

The solutions $P_l^m(x)$, called the associated Legendre polynomials, can be written down in terms of the derivatives of $P_l(x)$. Using the Rodrigue's formula for the latter, the solutions take the form,

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

for $l = 0, 1, \cdots, \infty$ and $m = -1, -l+1, \cdots, l-1, l$. They satisfy the orthogonality relation

$$\int_{-1}^{+1} P_{l'}^m(x) P_l^m(x) dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l'l}$$

4.5 Spherical Harmonics $Y_{lm}(\theta, \phi)$

As mentioned above, the two angular solutions of the Laplace equation are usually combined into spherical harmonics,

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)e^{im\phi} \tag{15}$$

Orthogonality:

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta Y_{l'm'}^*(\theta,\phi) Y_{lm}(\theta,\phi) = \delta_{l'l}\delta_{m'm}$$

Completeness:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta - \cos\theta')$$

The orthogonality and completeness of $Y_{lm}(\theta, \phi)$ imply that the angular part of any function can be expanded in terms of them. Thus, for any arbitrary function $f(r, \theta, \phi)$,

$$\boxed{f(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \hat{f}_{lm}(r) Y_{lm}(\theta,\phi)}$$

If f should satisfy the Laplace equation, then as we have seen, its radial dependence should have the form

$$\hat{f}_{lm}(r) = A_{lm}\, r^l + B_{lm}\, r^{-l-1}$$

Furthermore, spherical harmonics satisfy,

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{l,m}^*(\theta,\phi)$$

4.6 Application to Boundary-Value Problems

Consider the solution (14) of the Laplace equation in a region bounded by two spheres of radii $r = a$ and $r = b$. Let the potential on the spheres be specified as $\Phi(r = a, \theta, \phi) = V_a(\theta, \phi)$ and $\Phi(r = b, \theta, \phi) = V_b(\theta, \phi)$ Then, using the orthonormality of $Y_{lm}(\theta, \phi)$, the coefficients A_{lm} and B_{lm} can be determined in terms of the boundary conditions as,

$$\begin{aligned} A_{lm}\, a^l + B_{lm}\, a^{-l-1} &= \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta'\, Y^*_{lm}(\theta', \phi') V_a(\theta', \phi') \\ A_{lm}\, b^l + B_{lm}\, b^{-l-1} &= \int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta'\, Y^*_{lm}(\theta', \phi') V_b(\theta', \phi') \end{aligned}$$

Note that if either surface is moved to infinity with the condition that the potential on it remains finite, then $A_{lm} = 0$. Similarly, if either surface is moved to the origin with the condition that the potential on it does not blow up, then one gets $B_{lm} = 0$.

4.7 Boundary-Value Problem with Azimuthal Symmetry

From now on, our intention is to find a complicated expression for the simple function $1/|\vec{x} - \vec{x'}|$, for reasons that will become obvious.

Consider a problem with rotational symmetry about the z-axis. The solution to the potential problem then should not depend on the angle ϕ. Hence, its expansion only contains the $m = 0$ modes of (13),

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(\bar{A}_l\, r^l + \bar{B}_l\, r^{-l-1} \right) P_l(\cos\theta) \tag{16}$$

In a given problem, once the coefficients A_l and B_l are determined by the boundary conditions, then the above expression is the *unique* expansion of the potential. In particular this means that the way the r- and θ-dependent parts combine is unique. Hence, knowing only the r-dependence of the solution, its θ-dependence can be determined. For example, for $\theta = 0$, the potential can be expanded as a function of r (note that $P_l(1) = 1$),

$$\Phi(r, \theta = 0) = \sum_{l=0}^{\infty} \left(\bar{A}_l\, r^l + \bar{B}_l\, r^{-l-1} \right) \tag{17}$$

Then for non-zero θ, the θ-dependence can be inserted by multiplying each power of r^l and r^{-l-1} by $P_l(\cos\theta)$. Note that if $m \neq 0$, then the same power of r could correspond to different values of m and the uniqueness is lost.

Expansion of $\mathbf{1/|\vec{x} - \vec{x}'|}$: The funtion $1/|\vec{x} - \vec{x'}|$ can be regarded as the potential at $\vec{x'}$ due to a unit charge placed at $\vec{x}$, or *vice versa*. For $\vec{x} \neq \vec{x'}$, it satisfies the Laplace equation and hence can be expanded using the above techniques. Let us first expand this by brute force. Note that

$$\begin{aligned}
\frac{1}{|\vec{x}-\vec{x'}|} &= \frac{1}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}} \\
&= \frac{1}{r}\left(1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\gamma\right)^{-1/2} = \frac{1}{r'}\left(1+\frac{r^2}{r'^2}-2\frac{r}{r'}\cos\gamma\right)^{-1/2}
\end{aligned}$$

Here, γ is the angle between $\vec{x}$ and $\vec{x'}$. This can be expanded in powers of r'/r for $r > r'$ or in powers of r/r' for $r' > r$ (as the case may be) using the binomial expansion,

$$(a+b)^n = a^n b^0 + n a^{n-1} b^1 + \frac{n(n-1)}{2} a^{n-2} b^2 + \cdots$$

with $a = 1$. To treat both cases simultaneously, let us use the notation

$$r_< = \text{the smaller of } r \text{ and } r'$$

$$r_> = \text{the larger of } r \text{ and } r'$$

Then in either case,

$$\begin{aligned}
\frac{1}{|\vec{x}-\vec{x'}|} &= \frac{1}{r_>}\left(1+\frac{r_<^2}{r_>^2}-2\frac{r_<}{r_>}\cos\gamma\right)^{-1/2} \qquad (18)\\
&= \frac{1}{r_>}\Bigg[1+\frac{r_<}{r_>}\cos\gamma+\left(\frac{r_<}{r_>}\right)^2\frac{3\cos^2\gamma-1}{2} \\
&\qquad\qquad + \left(\frac{r_<}{r_>}\right)^3\frac{1}{2}(5\cos^3\gamma-3\cos\gamma)+\cdots\Bigg] \\
&= \frac{1}{r_>}\Bigg[1+\frac{r_<}{r_>}P_1(\cos\gamma)+\left(\frac{r_<}{r_>}\right)^2 P_2(\cos\gamma)+\left(\frac{r_<}{r_>}\right)^3 P_3(\cos\gamma)+\cdots\Bigg]
\end{aligned}$$

This suggests that the entire expansion, and not only the first few terms computed above, could be written in terms of the Legendre polynomials. To show this, we recognize that the problem of a point charge placed at $\vec{x}$ is invariant under rotations about the axis $\vec{x}/x$. Let us choose this to be the z direction. Then $\gamma = \theta$. When $\theta = 0$, the expansion has the form

$$\left.\frac{1}{|\vec{x}-\vec{x'}|}\right|_{\theta=0} = (r_>^2+r_<^2-2r_>r_<)^{-1/2} = (r_>-r_<)^{-1} = \frac{1}{r_>}(1-\frac{r_<}{r_>})^{-1} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}$$

For $\theta \neq 0$, the uniqueness of the expansion as discussed above means that each power of r^l or r^{-l-1} should be multiplied by $P_l(\cos\theta)$. As one can see, in the above expansion it does not matter whether the r in (16) or (17) is identified as $r_<$ or $r_>$. In either case one gets,

$$\boxed{\frac{1}{|\vec{x}-\vec{x'}|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)} \qquad (19)$$

Obviously, this equation remains valid even when $\vec{x}$ is not in the z-direction so that $\theta \neq \gamma$. Note that this expansion is valid when $r \neq r'$, as only then $1/|\vec{x}-\vec{x'}|$ satisfies the Laplace equation. In general, it satisfies the Poisson equation with a delta-function sourse supported at $\vec{x} = \vec{x'}$.

4.8 Addition Theorem for Spherical Harmonics

Let $\vec{x}$ and $\vec{x'}$ have spherical polar coordinates (r, θ, ϕ) and (r', θ', ϕ'), respectively. In terms of these, the relative angle γ between the two is given by

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$$

$|\vec{x} - \vec{x'}|$ depends on r, r' and the relative angle γ. To integrate over all directions of $\vec{x'}$, one option is to rotate the coordinate system such that the z-axis coincides with $\vec{x}$. However, if the integrand contains other functions of θ' and ϕ', *e.g.*, the boundary value of the potential or a charge density, then this procedure may complicate the expression.

The addition theorem for spherical harmonics allows us to factorize the dependence on the directions of $\vec{x}$ and $\vec{x'}$ in such situations. The theorem states that,

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi) \tag{20}$$

The proof of this theorem will not be given here. Substituting this in the expansion of $1/|\vec{x} - \vec{x'}|$, one gets,

$$\boxed{\frac{1}{|\vec{x} - \vec{x'}|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi)} \tag{21}$$

4.9 Expansion of Green Functions in Spherical Coordinates

As we have seen, when the boundary is specified at infinity, then the Green function is $G(\vec{x}, \vec{x'}) = 1/|\vec{x} - \vec{x'}|$ the expansion of which was obtained above.

Let us now consider the Green function for the exterior problem with a spherical boundary at radius $r = a$ on which the potential is specified. Using the method of images, we found this to be

$$G(\vec{x}, \vec{x'}) = \frac{1}{|\vec{x} - \vec{x'}|} + \frac{q'}{|\vec{x'} - \vec{y'}|}$$

where

$$q' = -\frac{a}{r}, \qquad y' = \frac{a^2}{r}$$

Both terms can be expanded using (21). As for the second term, note that $|\vec{x'}| > |\vec{y'}|$, and $\vec{y'}$ is parallel to $\vec{x}$. Then,

$$G(\vec{x}, \vec{x'}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right] Y^*_{lm}(\theta', \phi')Y_{lm}(\theta, \phi) \tag{22}$$

One can see that the behaviour of this functions changes at $r = r'$. Indeed, the radial part has the form,

$$\frac{1}{r^{l+1}}\left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right), \qquad \text{for} \qquad r' < r$$

$$\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\frac{1}{r'^{l+1}}, \qquad \text{for} \qquad r' > r$$

The behaviour has a natural interpretation if $G(\vec{x}, \vec{x'})$ is regarded as a potential at $\vec{x'}$ due to a unit charge at $\vec{x}$. Then, as r' increases beyond r, the potential due to the unit charge as well as the one due to the induced charge on the sphere keep on falling. But for $a < r' < r$, as the potential due to the unit charge decreses, the one due to the sphere increases, or vice-versa.

The reason for the change in behavious at $r = r'$ is obvious. The above expansions are obtained on the assumption that $G(\vec{x}, \vec{x'})$ satisfies the Laplace equation, which is not true at $r = r'$. Also note that the expansion satisfies other properties of $G(\vec{x}, \vec{x'})$ like symmetry in the arguments and vanishing at $r = a$ or $r' = a$.